Symmetric $\overline{X}$ Charts: Sensitivity to Nonnormality and Control-Limit Estimation

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Abstract

We study the classical symmetric $\overline{X}$-chart with control limits set k standard deviations from the known in-control mean. The standard deviation is estimated with in-control data, in what we refer to as Phase-I. We consider three performance measures: the average run length (ARL), the standard deviation of the conditional average run length (SDARL), and the corresponding coefficient of variation.

Modeling the $\overline{X}$ data as independent and identically distributed, with marginal distributions chosen from the Johnson family, we investigate in-control and out-of-control sensitivities to three factors: the third and fourth standardized moments of the $\overline{X}$ data distribution and the number of Phase-I observations. Considering both bounded and unbounded data distributions, our analytical, numerical, and Monte Carlo simulation results show that nonnormality has a substantial effect on all three performance measures; and the effects are nonmonotonic in both skewness and kurtosis.

We show that all three performance measures are flawed when estimating the standard deviation. In particular, we show that ARL and SDARL values increase, eventually becoming infinite, as the number of Phase-I observations decreases, even in cases where run length is finite with probability one. We show analytically that, for bounded data distributions with any finite shift, estimating the standard deviation sometimes results in infinite ARL and SDARL values.
Keywords: Average run length, control limits, Johnson family, kurtosis, skewness

1 Introduction

We study the effects of nonnormality and standard-deviation estimation on the *run-length properties* of $\overline{X}$ control charts, which are used to detect whether the mean of a data process is in or out of control (Montgomery 2013). Suppose that we have available an ostensibly independent and identically distributed (iid) sequence of observable sample means $\overline{X}_i$, $i = 1, 2, \ldots$, with known in-control mean μ_0. The $\overline{X}$ control-chart procedure monitors the sequence of sample means in order to detect a mean shift away from μ_0. An out-of-control signal is sent when a sample mean, say $\overline{X}_N$, first lies outside the chart's control limits. We focus on the case of symmetric control limits $\mu_0 \pm k\widehat{\sigma}_{\overline{X}}$, where k is a positive constant and $\widehat{\sigma}_{\overline{X}}$ is the estimated standard deviation of the in-control $\overline{X}$ process. We refer to $\mu_0 - k\widehat{\sigma}_{\overline{X}}$ as the lower control limit (LCL) and $\mu_0 + k\widehat{\sigma}_{\overline{X}}$ as the upper control limit (UCL). Following common usage, our numerical and Monte Carlo results assume that $k = 3$, although we provide some analytical results for general values of k.

1.1 Control-chart performance measures

Other than by economic criteria, control-chart performance is measured by the statistical properties of the random run lengths $N_\delta = \min\{i \in \mathbb{Z}^+ : |\overline{X}_i - \mu_0| > k\widehat{\sigma}_{\overline{X}}\}$ when the data-process mean is shifted from μ_0 by δ standard deviations $\sigma_{\overline{X}}$ of $\overline{X}$. (If the set $\{i \in \mathbb{Z}^+ : |\overline{X}_i - \mu_0| > k\widehat{\sigma}_{\overline{X}}\}$ is empty, then $N_\delta = \infty$.) Common performance measures include the average run length (ARL), denoted as $\mathrm{ARL}_\delta \equiv \mathrm{E}(N_\delta)$, and the standard deviation and percentiles of N_δ. For example, when the process is in control with mean μ_0, the resulting in-control ARL,

ARL_0, is preferred to be large, indicating a lower frequency of false alarms; when the process is out of control with the mean shift $\delta \neq 0$, ARL_δ is preferred to be small, indicating higher power in detecting the mean shift. For notational simplicity, we drop the subscript δ in N_δ and ARL_δ for the rest of this subsection.

Because we consider only iid data, for fixed LCL and UCL values, the distribution of N is geometric. Let $p(\widehat{\sigma}_{\overline{X}})$ denote the probability $\text{P}\{\overline{X} \notin \mu_0 \pm k\widehat{\sigma}_{\overline{X}}\}$, conditional on the Phase-I estimate of $\sigma_{\overline{X}}$. The mean of N is $1/p(\widehat{\sigma}_{\overline{X}})$ and the variance is $[1 - p(\widehat{\sigma}_{\overline{X}})]/[p(\widehat{\sigma}_{\overline{X}})]^2$. The LCL and UCL values can be fixed in two ways: (i) if $\sigma_{\overline{X}}$ has a known value (and therefore $\widehat{\sigma}_{\overline{X}} = \sigma_{\overline{X}}$) or (ii) if a particular realization of the estimated standard deviation $\widehat{\sigma}_{\overline{X}}$ is used.

Consider the general case, including the first case with an infinite Phase-I sample size m (i.e., $\widehat{\sigma}_{\overline{X}} = \sigma_{\overline{X}}$) and the second case with a finite m. Then,

$$\text{ARL} = \text{E}(N) = \text{E}_{\widehat{\sigma}_{\overline{X}}}\Big[\text{E}(N|\widehat{\sigma}_{\overline{X}})\Big] = \text{E}_{\widehat{\sigma}_{\overline{X}}}\Big[1/p(\widehat{\sigma}_{\overline{X}})\Big], \tag{1}$$

where the last equality is valid because, given a specific observation of $\widehat{\sigma}_{\overline{X}}$, the conditional distribution of N is geometric with the random probability of success $p(\widehat{\sigma}_{\overline{X}})$. The variance of the run length can be decomposed into two parts (e.g., Ross 2006, page 381):

$$\begin{aligned}
\text{Var}(N) &= \text{E}_{\widehat{\sigma}_{\overline{X}}}\Big[\text{Var}(N|\widehat{\sigma}_{\overline{X}})\Big] + \text{Var}_{\widehat{\sigma}_{\overline{X}}}\Big[\text{E}(N|\widehat{\sigma}_{\overline{X}})\Big] \\
&= \text{E}_{\widehat{\sigma}_{\overline{X}}}\Big[\big(1 - p(\widehat{\sigma}_{\overline{X}})\big)/\big(p(\widehat{\sigma}_{\overline{X}})\big)^2\Big] + \text{Var}_{\widehat{\sigma}_{\overline{X}}}\Big[1/p(\widehat{\sigma}_{\overline{X}})\Big] \\
&\equiv \text{E}_{\widehat{\sigma}_{\overline{X}}}\Big[\big(1 - p(\widehat{\sigma}_{\overline{X}})\big)/\big(p(\widehat{\sigma}_{\overline{X}})\big)^2\Big] + \text{SDARL}^2,
\end{aligned} \tag{2}$$

where SDARL denotes the standard deviation of the conditional ARL, $\text{E}(N|\widehat{\sigma}_{\overline{X}})$. Equation (2) shows that the variance of the run length is at least the squared SDARL.

For the first case ($\widehat{\sigma}_{\overline{X}}$ known), ARL and $\text{Var}(N)$ can be simplified as

$$\text{ARL} = 1/p(\sigma_{\overline{X}}) \quad \text{and} \quad \text{Var}(N) = [1 - p(\sigma_{\overline{X}})]/[p(\sigma_{\overline{X}})]^2$$

because $\widehat{\sigma}_{\overline{X}} = \sigma_{\overline{X}}$ and SDARL equals zero. For example and for later comparison, consider the special case where $\overline{X}$ has a normal distribution and the shift is $\delta = 0$ (that is, the mean is $\mu = \mu_0$). For the commonly used value $k = 3$, the in-control ARL is $\text{ARL}_0 = [2\Phi(-3)]^{-1} =$

$(.0027)^{-1} = 370.4$ and the standard deviation of the run length is $(1 - .0027)^{0.5}/.0027 = 369.9$, where $\Phi(\cdot)$ is the standard-normal cumulative distribution function (cdf).

The performance measures that we consider are ARL, SDARL, and CVARL, where

$$\text{CVARL} \equiv \text{SDARL}/\text{ARL}$$

is the coefficient of variation of the conditional ARL. These three performance measures are functions of the shift δ, the data process, and the quality of the estimated standard deviation; they are not functions of μ_0 or $\sigma_{\overline{X}}$. We are interested in the effects of both nonnormality and estimation error on these three performance measures. However, when only the nonnormality effect is discussed, we only consider the performance measure ARL because in this case, we assume that $\sigma_{\overline{X}}$ is known and hence, SDARL is zero.

Despite ARL being the most-popular performance measure, we also consider SDARL because some recent literature suggests using SDARL to quantify estimation errors. (See Section 2.2.) If the sample size for computing $\widehat{\sigma}_{\overline{X}}$ is small, the distribution of the conditional ARL has a long right tail and hence SDARL is large. Consequently, the value of ARL_0 for an observed $\widehat{\sigma}_{\overline{X}}$ might deviate highly from the nominal value, the ARL_0 for known $\sigma_{\overline{X}}$. We also consider CVARL, the normalized SDARL, since whether the value of SDARL is large depends on the value of ARL.

1.2 One-sided and two-sided charts

Our focus is on two-sided symmetric control charts, but lower- and upper-limit behaviors differ when $\delta \neq 0$. To better explain the ARL behavior of two-sided charts, we sometimes also consider lower one-sided charts and upper one-sided charts. We denote the corresponding ARLs by ARL_δ, ARL_δ^-, and ARL_δ^+, where the latter two correspond to run lengths until an $\overline{X}$ observation goes below the LCL and above the UCL, respectively. Each ARL value is the reciprocal of the corresponding tail probability. Let $p^- \equiv \text{P}(\overline{X} < \mu - k\widehat{\sigma}_{\overline{X}})$ denote the lower one-sided probability, $p^+ \equiv \text{P}(\overline{X} > \mu + k\widehat{\sigma}_{\overline{X}})$ denote the upper one-sided probability, and $p \equiv p^- + p^+$ denote the two-sided probability. Given specific LCL and UCL values, these probabilities can be obtained from the cdf of the corresponding data distribution and

$\mu = \mu_0 + \delta\sigma_{\overline{X}}$ value.

The behavior of the two-sided ARL_δ can be explained by ARL_δ^- and ARL_δ^+. Specifically, because $p = p^- + p^+$ and $\text{ARL}_\delta = 1/p$, $\text{ARL}_\delta^- = 1/p^-$, and $\text{ARL}_\delta^+ = 1/p^+$, we have

$$\text{ARL}_\delta = [(1/\text{ARL}_\delta^-) + (1/\text{ARL}_\delta^+)]^{-1}.$$

If both ARL_δ^- and ARL_δ^+ are infinite, then ARL_δ is infinite; if exactly one of ARL_δ^- or ARL_δ^+ is infinite, then ARL_δ equals the other. If the distribution of $\overline{X}$ is symmetric, then $\text{ARL}_0^- = \text{ARL}_0^+ = 2\text{ARL}_0$.

1.3 Assumptions and purpose

Three data-distribution assumptions are typically made: (i) the $\overline{X}$ process is iid; (ii) each $\overline{X}$ sample mean is normally distributed; and (iii) the value of $\sigma_{\overline{X}}$ is known. Of course, these assumptions are sometimes not true. For example, Pandit and Wu (1983) discuss the case of (dependent) time-series data and Montgomery (2013, Figures 8-9 and 8-10) discusses non-normal data. Unknown standard deviations, which need to be estimated, are also frequently encountered.

Our purpose is to relax the second and third assumptions to investigate the effects on ARL behavior. For purposes of relaxing Assumption (ii), we take the marginal distribution of $\overline{X}$ to be from the Johnson family, allowing us to study the effects of nonnormal skewness and kurtosis. (See Section 1.4 for background.) For relaxing Assumption (iii), we estimate $\sigma_{\overline{X}}$ using the sample standard deviation from a set of m previously obtained ("Phase-I") in-control $\overline{X}$ observations; we vary the value of m, with m going to infinity corresponding to knowing the value of $\sigma_{\overline{X}}$ with no error.

We make five Standing Assumptions. (a) The in-control mean μ_0 is known. (b) The control limits are symmetric about the in-control mean; that is, the control limits are $\mu_0 \pm k\sigma_{\overline{X}}$. (c) The control limits are $k = 3$ standard deviations $\sigma_{\overline{X}}$ from the mean. (d) When $\sigma_{\overline{X}}$ is estimated, the estimate is $\widehat{\sigma}_{\overline{X}} = S_{\overline{X}} \equiv [\sum_{i=1}^m (\overline{Y}_i - \mu_0)^2/m]^{1/2}$, where the $\overline{Y}$ observations are Phase-I $\overline{X}$ data. (e) The out-of-control data process differs from the in-control process by only a mean shift; that is, the out-of-control mean is $\mu = \mu_0 + \delta\sigma_{\overline{X}}$, and other process parameters do not change. (Notice that δ has no units.)

1.4 Johnson family and standardized moments

In our analyses and Monte Carlo experiments, the $\overline{X}$ data are iid with marginal distributions from the Johnson family of distributions (Johnson 1949; Johnson et al. 1994). Although we could have used other distribution families, the Johnson family is both tractable and general. The Johnson family can be partitioned into four subfamilies: the two-parameter normal family, three-parameter lognormal family, four-parameter bounded distributions, and four-parameter unbounded distributions. After fixing the mean and variance, both bounded and unbounded distributions have two parameters to determine distribution shape, which is often measured by (β_1, β_2), where $\beta_1 = \alpha_3^2$ is the squared skewness, $\beta_2 = \alpha_4$ is the kurtosis, and α_i is the ith standardized moment. Without duplication, the four subfamilies cover the entire feasible part (i.e., $\beta_2 \geq \beta_1 + 1$) of the (β_1, β_2) plane. All normal distributions lie at the point $(0, 3)$, and all lognormal distributions lie on a single infinite curve anchored at the normal distribution; the lognormal curve separates bounded and unbounded distributions, with bounded distributions having smaller kurtosis values. See, e.g., Dudewicz et al. (2004).

We use the Johnson family to model the higher-level $\overline{X}$ process rather than the lower-level, underlying X process, the latter sometimes being referred to as quality measurements. The two levels are related, of course, in that each $\overline{X}$ is an equally weighted average of n adjacent observations X. For iid data, the skewness and kurtosis of the $\overline{X}$ and X processes are related in the following well-known proposition.

Proposition 1 *(e.g., Johnson et al. 1994) For iid data $\{X_1, \ldots, X_n\}$ from a distribution lying at $(\beta_1(X), \beta_2(X))$, the sampling distribution of $\overline{X}$ with sample size n lies at $(\beta_1(\overline{X}), \beta_2(\overline{X}))$, where $\beta_1(\overline{X}) = \beta_1(X)/n$ and $\beta_2(\overline{X}) = 3 + (\beta_2(X) - 3)/n$.*

Proposition 1 is consistent with central limit theorems that say that sample means approach normality as n goes to infinity. For example, doubling the value of n moves the sampling distribution of $\overline{X}$ half-way to normality on the (β_1, β_2) plane. Proposition 1 implies that, whatever the distribution of X, the corresponding distribution of $\overline{X}$ lies at a point that satisfies $\beta_2(\overline{X}) \geq \beta_1(\overline{X}) + 3 - (2/n)$. A secondary implication is that $\overline{X}$ can attain all points on the feasible part of the (β_1, β_2) plane only for $n = 1$.

In the following sections, we consider only the process of the sample mean $\overline{X}$, seldom referring to the underlying data X. Therefore, we henceforth let α_3, β_1, and β_2 denote the skewness, squared skewness, and kurtosis of the marginal distribution of the sample mean $\overline{X}$. Because of Standing Assumption (e), these values apply to both in-control and out-of-control data processes. For clarity, however, we retain the $\overline{X}$ subscript in $\sigma_{\overline{X}}$ and $S_{\overline{X}}$.

Conducting the analysis based on the $\overline{X}$, rather than the X, data process has two advantages. First, multiple X processes can result in the same $\overline{X}$ process, so thinking in terms of $\overline{X}$ reduces the number of factors; in particular, the number n of observations of X that compose the sample mean $\overline{X}$ becomes implicit. Second, our results apply to any symmetric control chart, regardless of the control statistic used. For example, $\overline{X}$ could be replaced by a sample range R or a sample standard deviation S (except for situations in which the standard deviation of R or S is so large that the LCL is negative and adjusted to zero). Unlike $\overline{X}$, which becomes asymptotically normal when n is large, other statistics converge to normality slower. Nevertheless, in this paper, we present our results assuming that the control statistic is the sample average $\overline{X}$.

1.5 Contributions

Our results are as follows. ARL and SDARL values do not change monotonically with skewness and kurtosis. In addition, as the distribution shape moves away from normality, and for small values of m, the ARL and SDARL values sometimes change dramatically, often becoming infinite even when run lengths are finite with probability one. We provide various tables showing the relationships. The tables can be used to design an $\overline{X}$-chart: For any number of Phase-I observations m and any skewness and kurtosis of the sample means, one can determine the ARL and SDARL properties based on the results in this paper. We provide two new analytical results. Result 1 says that, when the data distribution is bounded, any Phase-I standard-deviation estimation guarantees infinite ARL values for certain modest values of k. Result 2 says that, for any symmetric unbounded Johnson distribution, the ARL (and therefore SDARL) values are infinite for all finite δ values when $m = 1$ degree of freedom is used to estimate the standard deviation. The implication of this result is that we can expect large, if not infinite, ARL and SDARL values for small m. In this sense, these performance measures are flawed.

1.6 Organization

The rest of this paper is organized as follows. Section 2 reviews the current literature on the effects of nonnormality and estimation for control charts. Section 3 shows that symmetric control limits are not appropriate for bounded distributions with finite m because ARL values are infinite in many practical settings. Section 4 illustrates the effects of nonnormality and estimation on the $\overline{X}$-chart performance. The asymptotic (infinite m) case of deterministic control limits is also discussed. Section 5 gives our conclusions.

2 Literature Review

Only one paper has considered the combined effect of nonnormality and estimation—Chen et al. (2008), which contains early results on which this paper is based. We review here papers on the effects of nonnormality alone and estimation alone.

2.1 Nonnormality effects

There is a good deal of literature concerning the effects of nonnormality on symmetric Shewhart $\overline{X}$ control charts with independent quality measurements X. Burrows (1962), Burr (1967), Schilling and Nelson (1976), Balakrishnan and Kocherlakota (1986), Chan et al. (1988), Borror et al. (1999), and Chen and Cheng (2007) discuss the nonnormality effects when the quality measurements in fact follow the Pearson-type, Burr (1942), two unimodal and two bimodal, symmetric Tukey-λ (Ramberg and Schmeiser 1972), Student's t, gamma, and Johnson distributions, respectively. All papers agree that nonnormality has significant effects on control-chart performance.

Several papers suggest using asymmetric control limits for nonnormal data. Five approaches are common: (i) symmetric probability limits, (ii) normal approximation via the central limit theorem with large n, (iii) transformation to normality, (iv) design for a specific known data distribution, and (v) split/weighted-variance methods. The first approach sets the control limits as percentiles so that the probabilities above the UCL and below the LCL are equal. Yourstone and Zimmer (1992) discuss such percentiles for the Burr distribution. Willemain and Runger (1996) estimate such percentiles from empirical reference distributions when the data distribution is unknown. For the second approach, Schilling and Nelson (1976) study how large the sample size n (within each $\overline{X}$ observation) of the $\overline{X}$ chart should be to obtain tail probabilities close to those for the normal distribution. Schader and Schmid (1989) and Ryan (1989) discuss a similar sample-size effect on the performance of p and np charts. The third approach involves the use of transformations. One kind is simple transformation, such as arcsin transformations for binomial data and square-root transformations for Poisson data (Ryan 1989 and Ryan and Schwertman 1997). A second kind is to transform a nonnormal random variable X to a standard-normal random variable via $\Phi^{-1}(F_X(X))$, where F_X is the cdf of X, such as the Q-chart method (Quesenberry 1991a,b,c). The fourth approach includes geometric midrange and range charts for monitoring the mean and variance of lognormal distributions (Ferrell 1958) and median, range, scale, and location charts for Weibull distributions (Nelson, 1979). For the fifth approach, Cowden (1957) proposes the split method, a heuristic which divides a nonsymmetric distribution at its mode into two supposedly normal distributions with the same mean (the original mode) and appropriately chosen (and likely different) standard deviations. One of the normals is used for the control chart's upper limit, and the other is used for the lower limit. A similar method is the weighted-variance technique of Choobineh and Ballard (1987). As an alternative to these five approaches for Shewhart charts, Borror et al. (1999) and Stoumbos and Reynolds (2000) show that exponentially weighted moving average charts are more robust to nonnormality than are $\overline{X}$ charts.

2.2 Estimation effects

Most literature on estimation effects assumes that the quality measurements X are iid normal with an unknown variance σ_X^2. Many papers assume that μ_0 is also unknown. All references cited later in this section estimate $\sigma_{\overline{X}}$ from m subgroups of n Phase-I quality measurements using $\widehat{\sigma}_X/\sqrt{n}$, where $\widehat{\sigma}_X$ is an estimate of σ_X. For a generic subgroup of n quality measurements, let R denote the range and S the sample standard deviation (dividing by $n-1$ when μ_0 is unknown and by n when known). Then choices for $\widehat{\sigma}_X$ include unbiased estimators $\overline{R}/d_{2,n}$,

$\overline{S}/c_{4,\nu_1+1}$, and $S_p/c_{4,\nu_2+1}$ and biased estimators S_p and $c_{4,\nu_2+1}S_p$, where $\overline{R}$, $\overline{S}$, and S_p are the average range, average sample standard deviation, and the square root of the pooled sample variance based on m subgroups (each of size n) of the Phase-I observations. The well-known bias-adjustment constants $d_{2,n}$ and $c_{4,\nu+1}$ are functions of the sample size n of each $\overline{X}$ observation (e.g., Montgomery 2013), where ν denotes the degrees of freedom that $\widehat{\sigma}_X$ has. The values of ν_1 and ν_2 are: $\nu_1 = n-1$ and $\nu_2 = m(n-1)$ when μ_0 is unknown and $\nu_1 = n$ and $\nu_2 = mn$ when known. Derman and Ross (1995) propose and show that S_p has a smaller mean square error (MSE) for estimating σ_X than the unbiased estimator $\overline{S}/c_{4,\nu_1+1}$. Vardeman (1999) later proposes the estimator $c_{4,\nu_2+1}S_p$ for its minimizing the MSE among all estimators γS_p, where $\gamma \in \mathbb{R}$. (Mahmoud et al. 2010 compare different estimators of σ_X for normal data; their results are consistent with Vardeman 1999.) Jensen et al. (2006) provide a literature review of estimation effects.

Perhaps the most-related work in the literature is Ghosh et al. (1981), who consider the effects of estimation for iid normal data with known in-control mean μ_0, a special case of the situation considered in this paper. They assume that σ_X is estimated with a χ statistic with ν degrees of freedom; for example, if $\widehat{\sigma}_X = S_p$, then $\nu = mn$ when μ_0 is known. Three of their results are relevant here. (i) As ν goes to infinity, the distribution of the run length converges to the geometric distribution, corresponding to the case of known control limits. (ii) Although the run length is finite with probability 1, if $k > \sqrt{\nu}$, then $\text{ARL}_\delta = \infty$ for all values of the shift δ. If $k = \sqrt{\nu}$, then $\text{ARL}_0 = \infty$ and ARL_δ is finite for $\delta \neq 0$. If $k < \sqrt{\nu}$, then ARL_δ is finite for all δ values. (iii) ARL_0 increases as ν decreases. Ng and Case (1992), Quesenberry (1993), Chen (1997), and Chakraborti (2000) also study the run-length distribution for the normal data case and obtain results consistent with Ghosh et al. (1981).

The control-chart performance measures used to study the estimation effect in most of the literature are the mean, standard deviation, and percentiles of the run length N_δ. Recent literature suggests using the chart performance metric SDARL because the distribution of the conditional ARL, $\text{E}(N_\delta|\widehat{\sigma}_{\overline{X}})$, is right-skewed. The SDARL reflects the variation in the conditional ARL from practitioner to practitioner with different observed values of $\widehat{\sigma}_{\overline{X}}$. Jones and Steiner (2012) first discuss the sample-size effect on SDARL for risk-adjusted CUSUM charts. Zhang et al. (2012), Zhang et al. (2013), Saleh et al. (2013), and Saleh et al. (2014) later use SDARL to discuss the sample-size effect on estimation errors for geometric, exponential CUSUM, exponentially weighted moving average, and Shewhart $\overline{X}$ and X control charts, respectively. They suggest using both ARL and SDARL, rather than just ARL, for choosing the value of the Phase-I sample size so as to have a high probability that the conditional ARL is close to the nominal value.

Some work compares choices of $\widehat{\sigma}_X$ based on the control-chart performance when μ_0 is unknown. Del Castillo (1996) shows that S_p is better than $\overline{S}/c_{4,n}$ in terms of false alarms rates and ability to detect shifts. Chen (1997), comparing the three unbiased estimators, shows that $S_p/c_{4,m(n-1)+1}$ and $\overline{R}/d_{2,n}$ have the (finite sample) standard deviations of the (unconditional) run length that are respectively closest and farthest to the limiting values with m infinite; but the same statement is not true for the ARL. Saleh et al. (2014), comparing the three unbiased estimators and the two biased estimators, show that the biased estimator $c_{4,m(n-1)+1}S_p$ is the best for the normal population in terms of having the smallest SDARL and an ARL_0 closest to the ARL_0 value with m infinite.

We consider only the estimator $\widehat{\sigma}_{\overline{X}} = S_{\overline{X}}$, defined in Section 1.3, when the value of μ_0 is known. We ignore the subgroup sample size n, so essentially $n = 1$ and therefore $S_{\overline{X}} = S_p$. Although our numerical results differ from those of other consistent estimators of $\sigma_{\overline{X}}$ (e.g., $S_p/c_{4,m+1}$ and $c_{4,m+1}S_p$), our results qualitatively generalize to those consistent estimators. Extending our conclusions beyond sensitivity analysis to chart design would require considering the specific estimator of $\sigma_{\overline{X}}$.

3 Bounded Distributions and Infinite ARL

In this section we examine properties of the run length N_δ when the $\overline{X}$ distribution is bounded, first with known standard deviation in Section 3.1 and then with estimated standard deviation in Section 3.2. In later sections, we focus on normal, lognormal, and unbounded distributions, although numerical results for bounded distributions are also given.

When the marginal distribution of $\overline{X}$ has bounded support, we show that ARL_δ may be infinite for both known and estimated control limits, even when δ is not zero. An implication of this section, then, is that symmetric control limits often are not appropriate when $\overline{X}$ has bounded support.

3.1 Known Standard Deviation

First, consider the situation in which $\sigma_{\overline{X}}$ is known; equivalently, $m = \infty$. The two-sided $\overline{X}$-chart control limits are now the constants $\mu_0 \pm k\sigma_{\overline{X}}$. We show that there are many situations for which ARL is infinite when the data are iid from a bounded distribution. For illustration, we compute ARL_δ values as a function of the skewness α_3 over the range of possible values $[-\sqrt{2}, \sqrt{2}]$ for the normal-distribution kurtosis value $\alpha_4 = \beta_2 = 3$. The sample means $\overline{X}$ then have either normal distributions (when $\alpha_3 = 0$) or Johnson bounded distributions (when $\alpha_3 \neq 0$).

The (unbounded) normal-distribution ARL_δ values are always finite; but when the distribution of $\overline{X}$ is bounded, the ARL_δ values are often infinite. All that is needed is $p = 0$, which occurs whenever the support of $\overline{X}$ lies entirely within the control limits, as illustrated in Table 1.

Table 1 gives ARL_δ^-, ARL_δ^+, and ARL_δ values for some examples for Johnson data with kurtosis $\beta_2 = 3$. Skewness values range over $\alpha_3 = 0$, ± 0.2, ± 0.4, ± 0.6, ± 0.8, ± 1, $\pm\sqrt{2}$, and shifts range over $\delta = 0, 0.5, 1, \infty$. Only non-negative δ values are shown; for any δ value, ARL_δ for the Johnson distribution with skewness α_3 equals $\text{ARL}_{-\delta}$ for the Johnson with skewness $-\alpha_3$ and the same kurtosis. For each entry, the ARL is the reciprocal of the tail probability, which is numerically computed from the Johnson cdf.

With Table 1 still in mind, consider some cases that illustrate the effects of skewness. For left-skewed distributions (i.e., $\alpha_3 < 0$), when α_3 is so negative that UCL is larger than the distribution's upper bound, then $\text{ARL}_\delta^+ = \infty$. (If the entire distribution is shifted to the right, however, then the infinity may be avoided. For example, when $\alpha_3 = -0.6$, then ARL_δ^+ is infinite for $\delta = 0$ but finite for $\delta = 1$.) When the skewness $\alpha_3 = -1$, the distribution's lower bound lies above the LCL; so ARL_δ^-, and hence ARL_δ, become infinite even for $\delta = 1$. Similarly, when the $\overline{X}$ distribution is very right skewed or the positive shift δ is so high that LCL is less than the distribution's lower bound, we have $\text{ARL}_\delta^- = \infty$.

In the normally distributed case, ARL_δ is monotonically decreasing in $|\delta|$. With bounded distributions, however, this monotonicity does not always hold. While it is true that ARL_δ^- is

Table 1: ARL_δ^-, ARL_δ^+, and ARL_δ for Johnson data with known standard deviation $\sigma_{\overline{X}}$, kurtosis $\beta_2 = 3$, skewness $\alpha_3 = 0$, ± 0.2, ± 0.4, ± 0.6, ± 0.8, ± 1, $\pm\sqrt{2}$, and shifts $\delta = 0, 0.5, 1, \infty$

	$\delta = 0$			$\delta = 0.5$			$\delta = 1$			$\delta = \infty$		
α_3	ARL_δ^-	ARL_δ^+	ARL_δ	ARL_δ^-	ARL_δ^+	ARL_δ	ARL_δ^-	ARL_δ^+	ARL_δ	ARL_δ^-	ARL_δ^+	ARL_δ
$-\sqrt{2}$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	1	1
-1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	1	1
-0.8	277	∞	277	12423	∞	12423	∞	∞	∞	∞	1	1
-0.6	241	∞	241	1323	∞	1323	17125	4731	3707	∞	1	1
-0.4	284	3.3E6	284	1183	3137	859	6523	123	120	∞	1	1
-0.2	397	3157	353	1701	335	280	8701	60	60	∞	1	1
0	741	741	370	4299	161	155	31574	44	44	∞	1	1
0.2	3157	397	353	58515	110	110	2.7E6	36	36	∞	1	1
0.4	3.3E6	284	284	5.7E21	85	85	∞	30	30	∞	1	1
0.6	∞	241	241	∞	69	69	∞	26	26	∞	1	1
0.8	∞	277	277	∞	57	57	∞	21	21	∞	1	1
1	∞	∞	∞	∞	53	53	∞	17	17	∞	1	1
$\sqrt{2}$	∞	∞	∞	∞	4.7	4.7	∞	4.7	4.7	∞	1	1

monotonically increasing in δ and that ARL_δ^+ is monotonically decreasing in δ, Table 1 shows that ARL_δ is not always monotonically decreasing in $|\delta|$; see, for example, the case $\alpha_3 = -0.4$. Nevertheless, as $|\delta|$ increases to infinity, ARL_δ approaches one, since eventually the entire bounded support of $\overline{X}$ lies outside the control limits.

The values of ARL_δ^-, ARL_δ^+, and ARL_δ change continuously with α_3, but are monotonic in neither α_3 nor $|\alpha_3|$. The reason is that the tail probabilities are not well explained by the third moment. Non-monotonic examples abound throughout Table 1. A persistent pattern in the table is that highly skewed distributions lead to infinite ARL_δ values; as discussed in the previous paragraph, however, $\text{ARL}_\delta = 1$ for large enough $|\delta|$ values.

3.2 Unknown Standard Deviation

Now consider the unknown $\sigma_{\overline{X}}$ case. As discussed in Section 1.3, estimation of the standard deviation is with m iid Phase-I observations $\overline{Y}$. Result 1 (below) says that when the data come from a bounded distribution, the Phase-I estimation causes ARL_δ to be infinite whenever k is not less than a certain value; specifically, ARL_0 is infinite whenever $k \geq 1$. Conditional on the estimate of $\sigma_{\overline{X}}$, the ARL is infinite whenever the random control limits lie beyond the distribution's support; therefore, the unconditional ARL is infinite whenever there is a positive probability that the estimated control limits enclose the distribution's support.

Result 1 *If the data are iid from any bounded distribution, then for every shift δ, the $\overline{X}$ chart with estimated control limits $\mu_0 \pm kS_{\overline{X}}$ has $\text{ARL}_\delta = \infty$ for every*

$$k \geq \frac{\max\{\mu_0 - (a + \delta\sigma_{\overline{X}}), (b + \delta\sigma_{\overline{X}}) - \mu_0\}}{\max\{\mu_0 - a, b - \mu_0\}}.$$

In particular, $\text{ARL}_0 = \infty$ for every $k \geq 1$.

Proof: Denote the support of the in-control distribution of $\overline{X}$ by $[a, b]$. Then $a \leq \mu_0 \leq b$, so the support of $S_{\overline{X}}$ is $[0, \max\{\mu_0 - a, b - \mu_0\}]$. The minimum occurs when all m data points are equal to μ_0 and the maximum occurs when all data points are at the end point farthest from μ_0. Let $\text{E}(N_\delta | S_{\overline{X}} = s)$ denote the conditional expected run length with estimated standard deviation s. Then $\text{E}(N_\delta | S_{\overline{X}} = s) = \infty$ whenever $\mu_0 - ks \leq a + \delta\sigma_{\overline{X}} < b + \delta\sigma_{\overline{X}} \leq \mu_0 + ks$, since then no data point can lie outside the estimated symmetric control limits. Rewriting the condition yields $s \geq \max\{\mu_0 - (a + \delta\sigma_{\overline{X}}), (b + \delta\sigma_{\overline{X}}) - \mu_0\}/k$. This event has positive probability whenever the upper limit $\max\{\mu_0 - a, b - \mu_0\}$ of the support of $S_{\overline{X}}$ is at least $\max\{\mu_0 - (a + \delta\sigma_{\overline{X}}), (b + \delta\sigma_{\overline{X}}) - \mu_0\}/k$. The inequality simplifies to obtain the result. □

4 The Effects of Nonnormality and Estimation

We now study ARL and SDARL values to investigate the effects of the $\overline{X}$ distribution shape and the Phase-I sample size when estimating $\sigma_{\overline{X}}$. Unlike the emphasis on infinite ARL values in Section 3, here we focus on computing specific ARL and SDARL values as functions of the mean shift δ, skewness α_3, kurtosis β_2, and Phase-I sample size m. Throughout this section, we take $k = 3$ and use symmetric charts. In Section 4.1 we consider the effect of nonnormality—over the entire skewness-kurtosis plane—when the standard deviation $\sigma_{\overline{X}}$ is known (and hence, SDARL is 0); consistent with earlier work, we show that ARL values are sensitive to distribution shape. Section 4.2 deals with the effect of estimating $\sigma_{\overline{X}}$ using m Phase-I observations; we observe that—for three symmetric unbounded distributions—ARL values increase as m decreases, empirically extending the normal-distribution result of Ghosh et al. (1981), and SDARL goes to zero as m goes to infinity. For the combined effects of distribution shape and Phase-I sample size, we show that—for the same three symmetric unbounded distributions—the estimation sensitivity (the effect of m on ARL_δ) is a complicated function of kurtosis, decreasing when δ is small and increasing when δ is large (e.g., 2).

4.1 Effects of Nonnormality

This section is concerned with the case in which we vary the distribution shape when the standard deviation $\sigma_{\overline{X}}$ is known. We numerically compute ARL_δ values as a function of the skewness α_3, kurtosis β_2, and shift δ. We consider points across the (skewness, kurtosis) plane. The results are shown in Figures 1, 2, and 3, which illustrate ARL_δ as a function of the kurtosis only, skewness only, and both skewness and kurtosis, respectively. As throughout this paper, the results assume the Johnson family of distributions.

4.1.1 Effects of kurtosis

Figure 1 considers the effect of (only) kurtosis by restricting attention to symmetric distributions, for which $\alpha_3 = 0$. The horizontal axis is the kurtosis β_2, from one to sixty; the vertical axis is the ARL_δ on a logarithmic scale. Five curves are shown, corresponding to $\delta = 0, 1, 2, 3, 6$ standard deviations. Only positive δ values are considered since, for symmetric distributions, $\text{ARL}_\delta = \text{ARL}_{-\delta}$. The ARL_δ curves are essentially flat to the right of $\beta_2 = 60$.

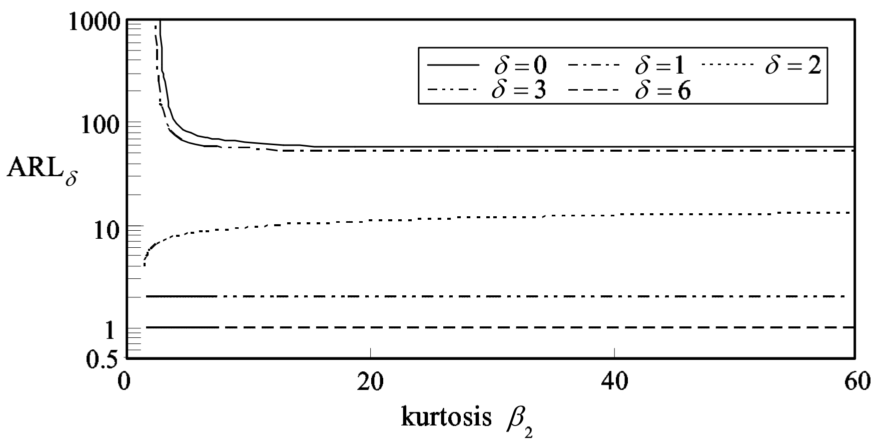

Figure 1: ARL_δ curves for symmetric distributions, as a function of the kurtosis β_2

Although the specific ARL values would change for different k or if we had chosen a family other than the Johnson, six patterns from Figure 1 are general. First, when $|\delta|$ is much less than k, ARL changes substantially with distribution shape for small values of the kurtosis. (Distribution shapes with kurtosis between 1 and 4 are widely different.) Second, for small values of $|\delta|$, ARL_δ goes to infinity as the kurtosis drops toward one. (Recall from Section 3.1 that $\text{ARL}_\delta = \infty$ whenever the support lies entirely within the control limits.) Third, as kurtosis increases, ARL_δ becomes insensitive to kurtosis. Fourth, ARL_δ is not always monotonic in the kurtosis; for example, ARL_0 has a minimum of about 57 close to $\beta_2 = 100$ (not shown in Figure 1). Fifth, for symmetric distributions, for $\delta = k = 3$, the probability of $\overline{X}$ falling outside the control limits is approximately 1/2, so $\text{ARL}_\delta \approx 2$; the deviation from 2 is the small-probability error that arises when $\overline{X}$ is outside the other tail's control limit. Sixth, as δ grows large (as illustrated by $\delta = 6$) ARL_δ goes to one for every k and for every distribution shape.

4.1.2 Effects of skewness

To study the effects of distribution skewness on ARL_δ values for unbounded distributions, we fix the kurtosis to $\beta_2 = 10$. The Johnson unbounded skewness values satisfy $\alpha_3 \in [-1.895, 1.895]$, where 1.895 is the (approximate) skewness for the lognormal distribution with $\beta_2 = 10$. Figure 2, which is analogous to Figure 1, shows the results. The horizontal axis gives the skewness values; the vertical axis contains the ARL_δ values. The four curves correspond to $\delta = 0, 0.5, 1, 1.5$ standard deviations of mean shift. Only positive values of δ are considered because, holding kurtosis constant, ARL_δ for skewness α_3 is equal to $\text{ARL}_{-\delta}$ for skewness $-\alpha_3$.

Like Figure 1, Figure 2 illustrates some patterns that hold more generally than for Johnson distributions. Here are three general results. First, for unbounded distributions with $\sigma_{\overline{X}}$ known, ARL_δ is finite. Second, for fixed skewness α_3, ARL_δ is not always decreasing in $|\delta|$, although as $|\delta|$ becomes large, ARL_δ decreases to one. Third, for fixed shift δ, ARL_δ is not always monotonic in α_3.

Table 2 gives ARL_δ^-, ARL_δ^+, and ARL_δ values for the same δ and many of the α_3 values as in Table 1, except that now $\beta_2 = 10$ and the distributions are unbounded. As with Table 1, Table 2 implies that symmetric control limits are not suitable for skewed distributions. Asymmetric control limits (e.g., setting the control limits so that the probabilities of falling below and above the limits are the same) seem to be more appropriate for skewed distributions, in that

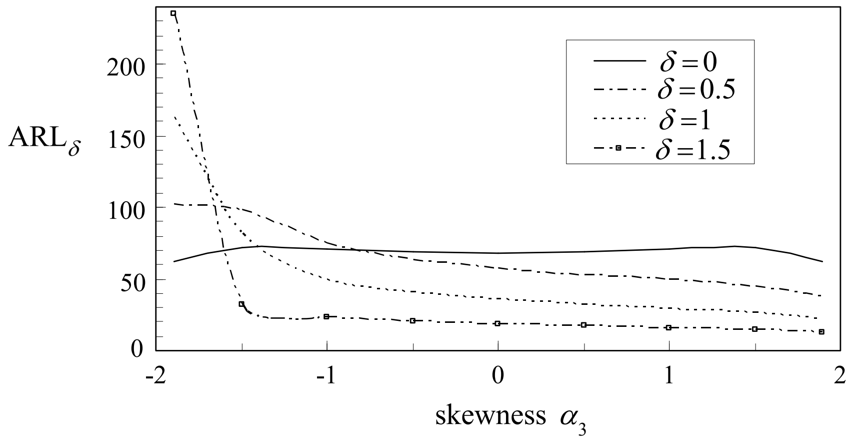

Figure 2: ARL_δ curves as a function of the skewness α_3 for unbounded Johnson distributions with kurtosis $\beta_2 = 10$

Table 2: ARL_δ^-, ARL_δ^+, and ARL_δ for Johnson data with kurtosis $\beta_2 = 10$, skewness $\alpha_3 = 0$, ± 0.5, ± 1, ± 1.5, ± 1.895, and shift $\delta = 0, 0.5, 1, 1.5$

	$\delta = 0$			$\delta = 0.5$			$\delta = 1$			$\delta = 1.5$		
α_3	ARL_δ^-	ARL_δ^+	ARL_δ	ARL_δ^-	ARL_δ^+	ARL_δ	ARL_δ^-	ARL_δ^+	ARL_δ	ARL_δ^-	ARL_δ^+	ARL_δ
−1.895	63	∞	63	102	∞	102	163	∞	163	257	2718	235
−1.5	75	1532	72	123	492	98	195	140	82	304	36	32
−1	91	323	71	150	148	75	240	63	50	373	25	23
−0.5	110	187	69	185	97	63	299	47	40	470	21	20
0	137	137	69	238	75	57	396	39	36	637	19	19
0.5	187	110	69	342	63	53	598	34	32	1003	18	17
1	323	91	71	658	53	49	1265	30	29	2313	16	16
1.5	1532	75	72	4277	45	44	10883	26	26	25606	15	15
1.895	∞	63	63	∞	38	38	∞	22	22	∞	13	13

we would likely regain the intuitively appealing property of decreasing ARL_δ as $|\delta|$ increases.

4.1.3 Effects of skewness and kurtosis

The (β_1, β_2) two-dimensional effects on ARL are further illustrated in Figure 3—the contour plot of ARL_0 on the (β_1, β_2) plane—with Subfigure (b) depicting a zoomed-in region (around the normal $(0, 3)$ point) of Subfigure (a). Figure 3 shows that ARL_0 changes substantially and non-monotonically when (β_1, β_2) deviates from $(0, 3)$, especially for bounded distributions, whose ARL_0 may be infinite (see Subfigure (b)). The approximate infinite-ARL_0 curve in Subfigure (b) lies below (but not far from) the true infinity curve; due to numerical errors in

Johnson-distribution fitting, the true infinity boundary is not known.

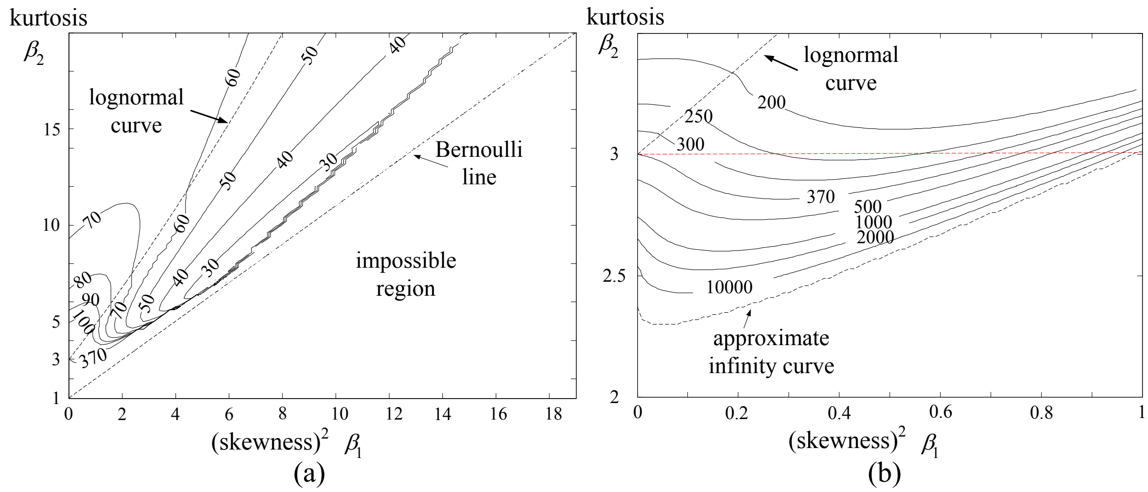

Figure 3: Contour plots of ARL_0 on the (β_1, β_2) plane for Johnson data with (a) $0 \le \beta_1 \le 19$ and $1 \le \beta_2 \le 20$ and (b) $0 \le \beta_1 \le 1$ and $2 \le \beta_2 \le 3.5$

4.2 Effects of Estimation

Next we study the case of unknown $\sigma_{\overline{X}}$, which is estimated by $S_{\overline{X}}$ from m Phase-I observations $\overline{Y}$. Since $S_{\overline{X}}$ is random, the control limits are random, resulting in run lengths with higher mean (and variance) than those of constant control limits.

We compute the values of ARL_δ and SDARL using Monte Carlo simulation and numerical integration. If simulation is used, the conditional means $\text{E}(N_\delta | S_{\overline{X}})$ are computed from all sets of generated Phase-I data; then the estimated ARL_δ and SDARL are the average and sample standard deviation of all realizations of $\text{E}(N_\delta | S_{\overline{X}})$, respectively. The standard errors are negligible, except for small values of m where the ARL and SDARL values approach infinity.

We consider symmetric Johnson data distributions with kurtosis values $\beta_2 = 3$, 5, 10, shift values $\delta = 0$, 1, 2, and all positive integers m. For clarity, let $\text{ARL}_\delta(m, \beta_2)$, $\text{SDARL}_\delta(m, \beta_2)$, and $\text{CVARL}_\delta(m, \beta_2)$ denote the average run length, SDARL, and CVARL, respectively, with shift δ and estimated control limits $\mu_0 \pm 3S_{\overline{X}}$ using m Phase-I observations from symmetric Johnson distributions with kurtosis β_2.

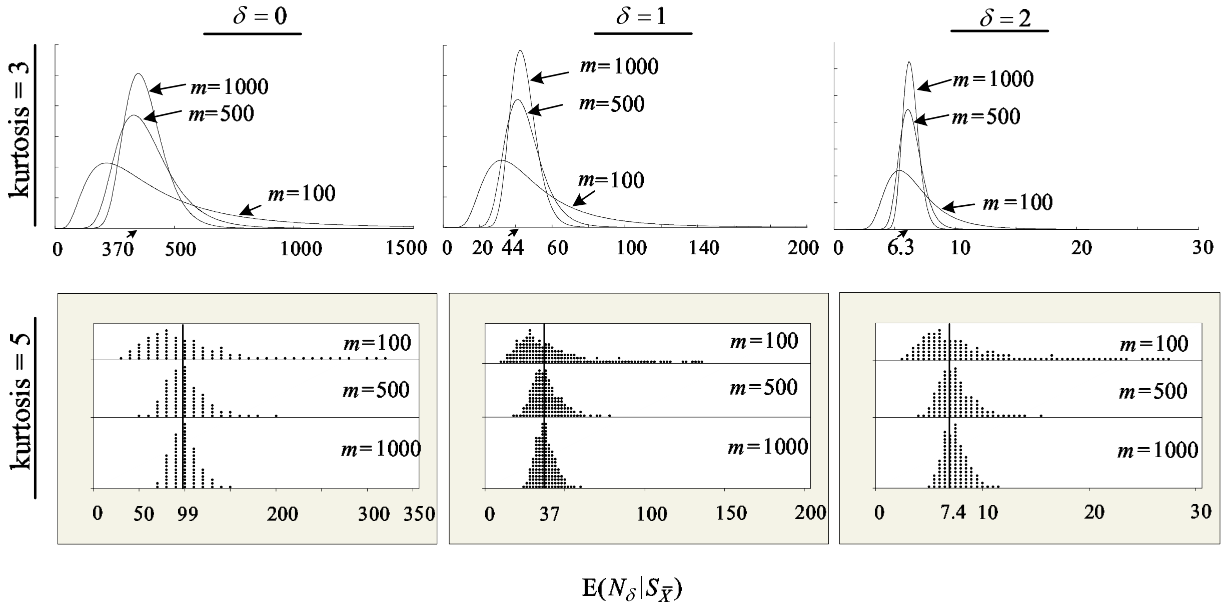

Figure 4: Density plots of the conditional mean $\mathrm{E}(N_\delta|S_{\overline{X}})$ for normal distributions ($\beta_2 = 3$) and dot plots for Johnson symmetric distributions with $\beta_2 = 5$

To gain insight, consider Figure 4, which gives density plots of the conditional mean $\mathrm{E}(N_\delta|S_{\overline{X}})$ for kurtosis 3 and dot plots for kurtosis $\beta_2 = 5$, as well as the corresponding values of $\mathrm{ARL}_\delta(\infty, \beta_2)$. The density plots were obtained using numerical integration, and the dot plots were obtained via Monte Carlo simulation. (The dot plots for $\beta_2 = 10$ are similar to those for $\beta_2 = 5$ and hence not shown here.) The ARL values are the means of these various conditional-distribution means. All distributions are right skewed, but much more so as m becomes smaller. As m increases, the more-symmetric distributions converge to the ARL_δ value for $m = \infty$. For a given number of Phase-I observations m, the distribution shapes do not differ much as a function of β_2, although their means do change.

For the three symmetric Johnson distributions, $\mathrm{ARL}_\delta(m, \beta_2)$ and $\mathrm{SDARL}_\delta(m, \beta_2)$ are illustrated in Figure 5 as functions of β_2, δ, and m. Table 3 contains detailed values used to create Figure 5, plus values for small m that do not appear in Figure 5, as well as the $\mathrm{CVARL}_\delta(m, \beta_2)$ values. The ARL values in Table 3 are reported to various precisions based on three rules: never are more than four digits reported, never are unknown (because of Monte Carlo sampling error) digits reported, and the hundredths digit is reported only when the ARL value is less than ten. The entries with an inequality are possibly infinity. The $\mathrm{SDARL}_\delta(m, \beta_2)$ and $\mathrm{CVARL}_\delta(m, \beta_2)$

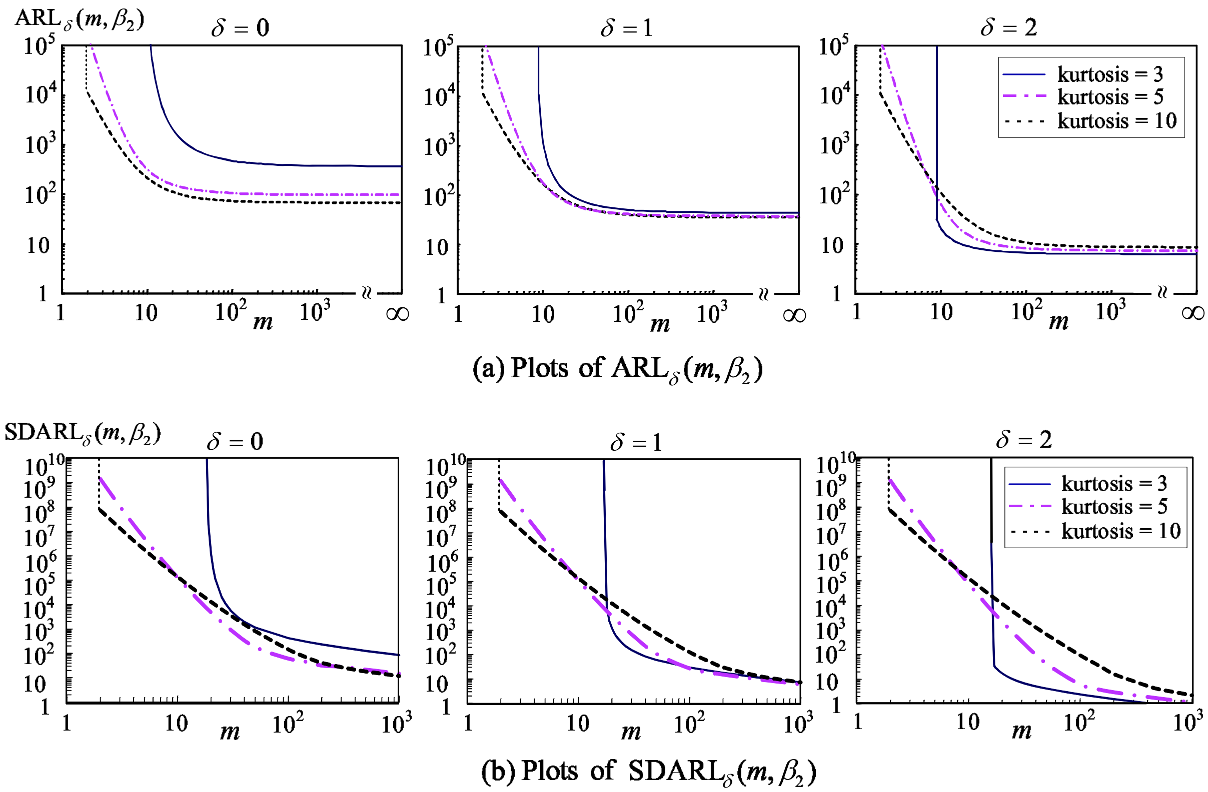

Figure 5: Plots of $\text{ARL}_\delta(m, \beta_2)$ and $\text{SDARL}_\delta(m, \beta_2)$ as functions of m for Johnson symmetric distributions

values (if finite) are reported to two digits and in scientific notation except for values less than 100. When both $\text{ARL}_\delta(\infty, \beta_2)$ and $\text{SDARL}_\delta(m, \beta_2)$ are infinite, $\text{CVARL}_\delta(m, \beta_2)$ is not meaningful and hence no value is reported.

For each value of β_2, two points emerge with respect to the estimation effect. First, as δ increases, both $\text{ARL}_\delta(m, \beta_2)$ and $\text{ARL}_\delta(\infty, \beta_2)$ go to one and hence, $\text{ARL}_\delta(m, \beta_2)$ approaches $\text{ARL}_\delta(\infty, \beta_2)$ (except for cases where $\text{ARL}_\delta(m, \beta_2) = \infty$). Second, as m becomes small, the $\text{ARL}_\delta(m, \beta_2)$ and $\text{CVARL}_\delta(m, \beta_2)$ values increase quickly (since $\text{SDARL}_\delta(m, \beta_2)$ increases more quickly than $\text{ARL}_\delta(m, \beta_2)$). This rapid increase in $\text{ARL}_\delta(m, \beta_2)$ is consistent with the results of Ghosh et al. (1981) (discussed in Section 2.2) for the normal distribution, where if $m < k^2 = 9$, then $\text{ARL}_\delta(m, \beta_2) = \infty$ for all δ values. (Notice that the $\text{ARL}_\delta(m, \beta_2)$ and squared $\text{SDARL}_\delta(m, \beta_2)$ are the first and second moments for the conditional ARL. If $\text{ARL}_\delta(m, \beta_2) = \infty$, then $\text{SDARL}_\delta(m, \beta_2) = \infty$.) The rapid increase also makes it difficult to determine $\text{ARL}_\delta(m, \beta_2)$ for small values of m. We report values obtained by Monte Carlo experimentation, but the true values depend on the extreme right tail of the distribution of $S_{\overline{X}}$; such realizations of $\widehat{\sigma}_{\overline{X}} = S_{\overline{X}}$ yield $p(\widehat{\sigma}_{\overline{X}})$ in Equation (1) quite close to zero. In fact, the Monte Carlo experiments cannot determine whether the true $\text{ARL}_\delta(m, \beta_2)$ and $\text{SDARL}_\delta(m, \beta_2)$ values are infinity or simply large. On the other hand, we can make a definite statement about the special case $m = 1$. Consistent with Ghosh et al. (1981) for normal data, our Result 2 shows analytically that $\text{ARL}_\delta(1, \beta_2) = \infty$ for unbounded symmetric Johnson data.

Result 2 *Consider the control-chart procedure with control limits $\mu_0 \pm kS_{\overline{X}}$ estimated from a single ($m = 1$) Phase-I observation and data from any unbounded symmetric Johnson distribution. The following three results hold.*

(a) *If $k < 1$, then $\text{ARL}_\delta < \infty$ for all δ.*

(b) *If $k > 1$, then $\text{ARL}_\delta = \infty$ for all δ.*

(c) *If $k = 1$, then $\text{ARL}_\delta = \infty$ for $\delta = 0$ and $\text{ARL}_\delta < \infty$ for all $\delta \neq 0$.*

The proof is in the Appendix.

That a bounded data distribution can lead to $\text{ARL}_\delta = \infty$, as stated in Result 1, is not surprising. The surprise is that, as stated in Result 2 for unbounded distributions, the use of estimation makes possible unconditional $\text{ARL}_\delta = \infty$, even while the ARL_δ values conditional on any realization of $\widehat{\sigma}_{\overline{X}}$ are finite with probability one for unbounded distributions.

Figure 5 and Table 3 also show the simultaneous effects of nonnormality and estimation. The substantial nonnormality effects on the control-chart performance with estimated control limits are similar to those with known control limits as shown in Figure 1. In particular, the curves in Figure 1 being convex or concave correspond to the curves' order in Figure 5. Estimation sensitivity is a complicated function of kurtosis, decreasing when δ is small and increasing when δ is large (e.g., 2).

5 Summary and Conclusions

Performance-measure values ARL, SDARL, and CVARL for symmetric $\overline{X}$ control charts are highly sensitive to both nonnormality and estimation error, as measured by skewness and kurtosis for nonnormality and by the number of Phase-I observations for estimation error. Our nonnormality results show that all three performance-measure values behave in nonmonotonic, and therefore nonintuitive, ways when the data distribution is asymmetric. To regain monotonicity, asymmetric limits that mimic the data distribution seem reasonable. Our estimation results show that—for performance-measure values to be close to the performance-measure values when the standard deviation is known—the number of Phase-I observations needs to be (maybe surprisingly) large.

First we considered nonnormality, providing analytical and numerical results for ARL. For bounded distributions, the ARL_δ values can be infinite when the run length itself is infinite. For unbounded distributions, the ARL_δ values are always finite if the standard deviation of $\overline{X}$ is finite. When finite, the ARL_δ values can change nonmonotonically with both skewness and kurtosis. Practitioners need to be aware that nonnormality can dramatically affect ARL_δ values.

Second, we considered estimation error, providing analytical, numerical, and Monte Carlo results. For estimated standard deviation of bounded data distributions, we show in Result 1 (where we relax the assumption that the control limits are three standard deviations from the mean) that ARLs are infinite whenever k is not less than a certain value. Specifically, in-control ARLs are infinite when the control-chart limits are at least one estimated standard deviation from the center line. For unbounded data distributions, our tables and figures illustrate the sensitivity to the number m of Phase-I observations; the ARL_δ and/or SDARL values can be infinite when m is small. In our Result 2 (again k not confined to 3), we extend the normality results of Ghosh et al. (1981) to symmetric unbounded Johnson data, but only for the special case of a single Phase-I observation, implying that all ARL_δ values go to infinity as the number of Phase-I observations decreases to $m = 1$ (for all $k > 1$).

An implication is that all three performance measures—ARL, SDARL, and CVARL—are flawed. Although the probability of an infinite run length is zero, all three performance measures become infinite relatively easily, while being invisible to every practitioner. In practice, infinite ARL_0 is good only when the corresponding ARL_δ is small for $\delta \neq 0$. However, when ARL_0 is infinite, ARL_δ is usually infinite for all values of δ or huge for small values of $|\delta|$. Moreover, for normal data and any finite δ, ARL_δ, and hence SDARL, are infinite for all $m < k^2$. Any adequate performance measure should indicate that larger values of m are better.

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Appendix: Proof of Result 2

When $m = 1$, there is only one Phase-I observation, denoted by $\overline{Y}$. In this simple case, the standard-deviation estimate is $S_{\overline{X}} = |\overline{Y}|$. The in-control data follow an unbounded symmetric Johnson distribution with location parameter $\xi = \mu_0 = 0$, scale parameter $\lambda > 0$, and shape parameters $\gamma = 0$ (for a symmetric distribution) and $\phi > 0$. Since ARL_δ is functionally independent of the standard deviation, without loss of generality set $\lambda = 1$. The probability density function (pdf) of $S_{\overline{X}}$ is then, for $s > 0$,

$$f_{S_{\overline{X}}}(s) = 2f_{\overline{Y}}(s) = \phi\sqrt{\frac{2}{\pi(s^2+1)}}\exp\Big\{-\frac{1}{2}\Big[\phi\ln(s+\sqrt{s^2+1}\,)\Big]^2\Big\},$$

where $f_{\overline{Y}}(\cdot)$ is the pdf of $\overline{Y}$ (Johnson et al. 1994).

Let N, N_U, and N_L denote the run lengths for two-sided, upper one-sided, and lower one-sided $\overline{X}$ charts with control limits $\mu_0 \pm kS_{\overline{X}}$, $\mu_0 + kS_{\overline{X}}$, and $\mu_0 - kS_{\overline{X}}$, respectively. Since $N = \min(N_U, N_L)$, every realization of N is no larger than N_U or N_L and hence, $\text{E}(N) \leq \min(\text{E}(N_U), \text{E}(N_L))$. Furthermore, because of symmetry, $\text{E}(N_U)$ with mean shift δ is the same as $\text{E}(N_L)$ with mean shift $-\delta$. When $\delta = 0$, $\text{E}(N) = 0.5\text{E}(N_U) = 0.5\text{E}(N_L)$.

To prove Result 2, we first prove the following results for $\text{E}(N_U)$, valid for all finite δ:

(a) If $k < 1$, then $\text{E}(N_U) < \infty$.

(b) If $k > 1$, then $\text{E}(N_U) = \infty$.

(c) If $k = 1$, then $\text{E}(N_U) = \infty$ for $\delta \leq 0$ and $\text{E}(N_U) < \infty$ for $\delta > 0$.

The proofs for these $\text{E}(N_U)$ results proceed as follows. Using the definition of N_U, taking the expected value of all realizations of the estimated standard deviation, setting $\mu_0 = 0$, transforming from an in-control unbounded Johnson random variable V to a standard normal random variable $Z = \phi\ln(V + \sqrt{V^2+1})$, and expressing the expected value as an integral using the pdf of $S_{\overline{X}}$ yields

$$\begin{aligned}
\mathrm{E}(N_U) &= \mathrm{E}_{S_{\overline{X}}}\Big[\frac{1}{P\{\overline{X} > \mu_0 + kS_{\overline{X}}\}}\ \Big|\ S_{\overline{X}}\Big] = \mathrm{E}_{S_{\overline{X}}}\Big[\frac{1}{P\{\overline{X} > kS_{\overline{X}}\}}\ \Big|\ S_{\overline{X}}\Big] \\
&= \mathrm{E}_{S_{\overline{X}}}\Big[\frac{1}{P\{V > kS_{\overline{X}} - \delta\}}\ \Big|\ S_{\overline{X}}\Big] \\
&= \mathrm{E}_{S_{\overline{X}}}\Big[\frac{1}{P\{Z > \phi\ln\big(kS_{\overline{X}} - \delta + \sqrt{(kS_{\overline{X}} - \delta)^2 + 1}\ \big)\}}\ \Big|\ S_{\overline{X}}\Big] \\
&= \mathrm{E}_{S_{\overline{X}}}\left[\frac{1}{1 - \Phi\Big[\phi\ln\big(kS_{\overline{X}} - \delta + \sqrt{(kS_{\overline{X}} - \delta)^2 + 1}\ \big)\ \Big]}\ \middle|\ S_{\overline{X}}\right] \\
&= \sqrt{\frac{2}{\pi}}\phi\int_0^{\infty}\frac{1}{[1 - \Phi(b)]\sqrt{s^2 + 1}}\ \exp\Big\{-\frac{1}{2}\Big[\phi\ln(s + \sqrt{s^2 + 1}\ )\Big]^2\Big\}\ ds,
\end{aligned}$$

where $b \equiv \phi\ln\big(ks - \delta + \sqrt{(ks - \delta)^2 + 1}\ \big)$. By Birnbaum (1942) and Pollak (1957), the bounds for a standard-normal upper-tail probability $1 - \Phi(t)$ are

$$\sqrt{\frac{2}{\pi}}\ \frac{e^{-\frac{t^2}{2}}}{t + \sqrt{t^2 + 4}} < 1 - \Phi(t) < \sqrt{\frac{2}{\pi}}\ \frac{e^{-\frac{t^2}{2}}}{t + \sqrt{t^2 + \frac{8}{\pi}}}, \qquad t > 0.$$

We now prove part (a), where $k < 1$. If $\delta \le 0$, we have $ks - \delta + \sqrt{(ks - \delta)^2 + 1} > 1$ and hence $b > 0$. Therefore,

$$\begin{aligned}
\mathrm{E}(N_U) &\le \phi\int_0^{\infty}\frac{b + \sqrt{b^2 + 4}}{\sqrt{s^2 + 1}}\ \exp\left\{\frac{1}{2}\Big[b^2 - \big[\phi\ln(s + \sqrt{s^2 + 1}\ )\big]^2\Big]\right\}\ ds \\
&= \phi\int_0^{\infty}\frac{\phi\ln\big(ks - \delta + \sqrt{(ks - \delta)^2 + 1}\ \big) + \sqrt{\big[\phi\ln\big(ks - \delta + \sqrt{(ks - \delta)^2 + 1}\ \big)\big]^2 + 4}}{\sqrt{s^2 + 1}} \\
&\qquad \exp\left\{\frac{1}{2}\Big[\big[\phi\ln\big(ks - \delta + \sqrt{(ks - \delta)^2 + 1}\ \big)\big]^2 - \big[\phi\ln(s + \sqrt{s^2 + 1}\ )\big]^2\Big]\right\}\ ds \\
&< \infty.
\end{aligned}$$

In the above equation, the upper bound of $\mathrm{E}(N_U)$ is finite because for all $s > -\delta/(1 - k)$, $0 < ks - \delta < s$ and hence, $0 < \ln\big(ks - \delta + \sqrt{(ks - \delta)^2 + 1}\ \big) < \ln(s + \sqrt{s^2 + 1})$.

If $\delta > 0$, we have $b > 0$ for all $s > \delta/k$. Therefore,

$$
\begin{aligned}
\mathrm{E}(N_U) &= \sqrt{\frac{2}{\pi}}\phi \int_0^{\delta/k} \frac{1}{[1-\Phi(b)]\sqrt{s^2+1}} \exp\Big\{-\frac{1}{2}\Big[\phi \ln(s+\sqrt{s^2+1}\,)\Big]^2\Big\}\, ds \\
&\quad + \sqrt{\frac{2}{\pi}}\phi \int_{\delta/k}^{\infty} \frac{1}{[1-\Phi(b)]\sqrt{s^2+1}} \exp\Big\{-\frac{1}{2}\Big[\phi \ln(s+\sqrt{s^2+1}\,)\Big]^2\Big\}\, ds \\
&< 2 + \sqrt{\frac{2}{\pi}}\phi \int_{\delta/k}^{\infty} \frac{1}{[1-\Phi(b)]\sqrt{s^2+1}} \exp\Big\{-\frac{1}{2}\Big[\phi \ln(s+\sqrt{s^2+1}\,)\Big]^2\Big\}\, ds \\
&< \infty.
\end{aligned}
$$

In the above equation, the first inequality holds because $1-\Phi(b) > 0.5$ for $b < 0$. The second inequality holds because $ks - \delta < s$ for $k < 1$ and $\delta > 0$, and the proof is similar to that for the case of $k < 1$ and $\delta \le 0$.

In part (b), we consider the case of $k > 1$. For any finite δ,

$$
\begin{aligned}
\mathrm{E}(N_U) &\ge \phi \int_0^{\infty} \frac{\phi \ln\big(ks-\delta+\sqrt{(ks-\delta)^2+1}\,\big) + \sqrt{\big[\phi \ln\big(ks-\delta+\sqrt{(ks-\delta)^2+1}\,\big)\big]^2 + 8/\pi}}{\sqrt{s^2+1}} \\
&\qquad \exp\Big\{\frac{1}{2}\Big[\big[\phi \ln\big(ks-\delta+\sqrt{(ks-\delta)^2+1}\,\big)\big]^2 - \big[\phi \ln(s+\sqrt{s^2+1}\,)\big]^2\Big]\Big\}\, ds \\
&\ge \phi \int_{\max\{0,\delta/(k-1)\}}^{\infty} (s^2+1)^{-0.5}\Big\{\phi \ln\big(ks-\delta+\sqrt{(ks-\delta)^2+1}\,\big) + \\
&\qquad \sqrt{\big[\phi \ln\big(ks-\delta+\sqrt{(ks-\delta)^2+1}\,\big)\big]^2 + 8/\pi}\Big\} \\
&\qquad \exp\Big\{\frac{1}{2}\Big[\big[\phi \ln\big(ks-\delta+\sqrt{(ks-\delta)^2+1}\,\big)\big]^2 - \big[\phi \ln(s+\sqrt{s^2+1}\,)\big]^2\Big]\Big\}\, ds.
\end{aligned}
$$

In the last inequality, the right-hand side is infinite because for all $s > \delta/(k-1)$, $\ln\big(ks-\delta+\sqrt{(ks-\delta)^2+1}\,\big) > \ln(s+\sqrt{s^2+1}) > 0$. Hence, when $k > 1$, $\mathrm{E}(N_U) = \infty$ for any finite δ.

In part (c), we consider the case of $k = 1$. If $\delta \le 0$, then

$$\begin{aligned}
\mathrm{E}(N_U) &= \sqrt{\frac{2}{\pi}}\phi \int_0^\infty \frac{1}{1-\Phi\big[\phi\ln\big(s-\delta+\sqrt{(s-\delta)^2+1}\,\big)\big]}\cdot\frac{1}{\sqrt{s^2+1}} \\
&\qquad \exp\Big\{-\frac{1}{2}\Big[\phi\ln(s+\sqrt{s^2+1}\,)\Big]^2\Big\}\,ds, \\
&\ge \phi\int_0^\infty \frac{\phi\ln\big(s-\delta+\sqrt{(s-\delta)^2+1}\,\big)+\sqrt{\big[\phi\ln\big(s-\delta+\sqrt{(s-\delta)^2+1}\,\big)\big]^2+8/\pi}}{\sqrt{s^2+1}} \\
&\qquad \exp\bigg\{\frac{1}{2}\Big[\big[\phi\ln\big(s-\delta+\sqrt{(s-\delta)^2+1}\,\big)\big]^2-\big[\phi\ln(s+\sqrt{s^2+1}\,)\big]^2\Big]\bigg\}\,ds.
\end{aligned}$$

If $\delta < 0$, the right-hand side of the last inequality is infinity and hence, $\mathrm{E}(N_U) = \infty$. If $\delta = 0$,

$$\begin{aligned}
\mathrm{E}(N_U) &\ge \phi\int_0^\infty \frac{\phi\ln\big(s+\sqrt{s^2+1}\,\big)+\sqrt{\big[\phi\ln\big(s+\sqrt{s^2+1}\,\big)\big]^2+8/\pi}}{\sqrt{s^2+1}}\,ds \\
&\ge \phi\int_0^\infty \frac{1}{\sqrt{s^2+1}}\,ds = \infty.
\end{aligned}$$

Therefore, for $k = 1$, if $\delta \le 0$, then $\mathrm{E}(N_U) = \infty$. If $\delta > 0$, we can prove that $\mathrm{E}(N_U) < \infty$. The proof is similar to that in part (b) for $\delta > 0$.

Using the results for $\mathrm{E}(N_U)$, we have the following results for $\mathrm{E}(N_L)$, valid for all finite δ:

(a) If $k < 1$, then $\mathrm{E}(N_L) < \infty$.

(b) If $k > 1$, then $\mathrm{E}(N_L) = \infty$.

(c) If $k = 1$, then $\mathrm{E}(N_L) = \infty$ for $\delta \ge 0$ and $\mathrm{E}(N_L) < \infty$ for $\delta < 0$.

Since $\mathrm{E}(N) \le \min(\mathrm{E}(N_U), \mathrm{E}(N_L))$, the proof for Result 2 is complete by combining the results for $\mathrm{E}(N_U)$ and $\mathrm{E}(N_L)$. □

Table 3: $\text{ARL}_\delta(m, \beta_2)$, $\text{SDARL}_\delta(m, \beta_2)$, and $\text{CVARL}_\delta(m, \beta_2)$ values for symmetric Johnson data distributions as functions of kurtosis β_2, shift δ, and number of Phase-I observations m

		$\beta_2 = 3$			$\beta_2 = 5$			$\beta_2 = 10$		
δ	m	ARL	SDARL	CVARL	ARL	SDARL	CVARL	ARL	SDARL	CVARL
	1	∞	∞		∞	∞		∞	∞	
	2	∞	∞		1.6E5	1.4E9	8.6E3	1.1E4	7.5E7	6.7E3
	3	∞	∞		2.0E4	1.1E8	5.5E3	3.1E3	1.4E7	4.4E3
	4	∞	∞		5.2E3	1.9E7	3.7E3	1.4E3	4.3E6	3.1E3
	5	∞	∞		2.1E3	5.4E6	2.5E3	8.0E2	1.8E6	2.3E3
	6	∞	∞		1.1E3	1.9E6	1.7E3	5.3E2	9.0E5	1.7E3
	7	∞	∞		7.1E2	8.2E5	1.2E3	3.9E2	5.0E5	1.3E3
	8	∞	∞		5.1E2	4.1E5	8.0E2	3.1E2	3.1E5	1.0E3
	9	∞	∞		3.9E2	2.2E5	5.6E2	2.5E2	2.0E5	7.9E2
	10	1.174E6	$\geq$ 1E137	$\geq$ 9E130	325	1.3E5	3.9E2	218	1.4E5	6.3E2
	11	1.039E5	$\geq$ 9E110	$\geq$ 9E105	280	8.0E4	2.8E2	192	9.8E4	5.1E2
0	12	3.103E4	$\geq$ 2E89	$\geq$ 7E84	249	5.2E4	2.1E2	174	7.2E4	4.2E2
	15	5,888	$\geq$ 1E42	$\geq$ 2E38	196	1.7E4	89	139	3.4E4	2.4E2
	20	2,090	1.2E6	5.7E2	158.9	4.8E3	30	112.7	1.3E4	1.2E2
	30	1,005	5.9E3	5.9	132.7	9.4E2	7.1	93.1	3.6E3	39
	50	637.3	1.2E3	1.8	116.8	1.8E2	1.6	81.3	8.3E2	10
	100	477.4	4.3E2	.89	107.0	61	.57	74.2	1.4E2	1.9
	150	437.2	2.9E2	.66	104.1	44	.43	72.2	65	.90
	200	418.9	2.3E2	.55	102.7	32	.31	71.2	41	.57
	300	401.7	1.7E2	.43	101.3	29	.28	70.3	27	.39
	400	393.5	1.4E2	.37	100.6	24	.24	69.8	21	.31
	500	388.7	1.3E2	.33	100.2	21	.21	69.5	18	.26
	1000	379.4	85	.22	99.4	15	.15	69.0	12	.17
	∞	370.4	0	0	98.6	0	0	68.5	0	0
	1	∞	∞		∞	∞		∞	∞	
	2	∞	∞		1.5E5	1.4E9	8.9E3	1.1E4	7.4E7	6.8E3
	3	∞	∞		1.8E4	1.0E8	5.8E3	3.0E3	1.4E7	4.5E3
	4	∞	∞		4.4E3	1.8E7	4.1E3	1.3E3	4.2E6	3.3E3
	5	∞	∞		1.7E3	4.9E6	3.0E3	7.3E2	1.8E6	2.5E3
	6	∞	∞		8.1E2	1.8E6	2.2E3	4.7E2	8.9E5	1.9E3
	7	∞	∞		4.7E2	7.4E5	1.6E3	3.3E2	4.9E5	1.5E3
	8	∞	∞		3.2E2	3.6E5	1.1E3	2.5E2	3.0E5	1.2E3
	9	1.030E4	$\geq$ 1E151	$\geq$ 1E147	2.3E2	1.9E5	8.4E2	2.0E2	2.0E5	9.8E2
	10	1,276	$\geq$ 4E121	$\geq$ 3E118	179	1.1E5	6.2E2	170	1.3E5	7.9E2
	11	571.6	$\geq$ 2E96	$\geq$ 3E93	147	6.9E4	4.7E2	146	9.7E4	6.6E2
1	12	357.9	$\geq$ 2E75	$\geq$ 5E72	126	4.5E4	3.5E2	128	7.1E4	5.6E2
	15	174.6	$\geq$ 5E29	$\geq$ 3E27	91	1.4E4	1.6E2	97	3.3E4	3.4E2
	20	107.0	1.2E3	11	68.8	3.8E3	56	72.9	1.3E4	1.7E2
	30	74.2	1.6E2	2.1	54.2	6.9E2	13	55.7	3.5E3	63
	50	58.6	61	1.0	46.0	1.1E2	2.4	45.7	7.8E2	17
	100	50.3	30	.59	41.3	27	.66	40.0	1.3E2	3.2
	150	48.0	22	.46	39.9	19	.47	38.4	53	1.4
	200	46.9	18	.38	39.2	15	.39	37.6	30	.80
	300	45.9	14	.31	38.6	12	.30	36.9	19	.51
	400	45.4	12	.26	38.2	9.9	.26	36.5	14	.38
	500	45.1	10	.23	38.1	8.8	.23	36.3	11	.32
	1000	44.5	7.2	.16	37.7	6.0	.16	35.9	7.3	.20
	∞	44.0	0	0	37.3	0	0	35.5	0	0

Continued on next page

Table 3 – continued from previous page

		$\beta_2 = 3$			$\beta_2 = 5$			$\beta_2 = 10$		
δ	m	ARL	SDARL	CVARL	ARL	SDARL	CVARL	ARL	SDARL	CVARL
	1	∞	∞		∞	∞		∞	∞	
	2	∞	∞		1.3E5	1.2E9	9.5E3	1.0E4	7.4E7	7.0E3
	3	∞	∞		1.4E4	8.8E7	6.5E3	2.8E3	1.4E7	4.8E3
	4	∞	∞		3.0E3	1.5E7	4.9E3	1.2E3	4.2E6	3.6E3
	5	∞	∞		1.0E3	3.9E6	3.8E3	6.2E2	1.8E6	2.8E3
	6	∞	∞		4.5E2	1.4E6	3.0E3	3.8E2	8.7E5	2.3E3
	7	∞	∞		2.4E2	5.5E5	2.4E3	2.5E2	4.8E5	1.9E3
	8	∞	∞		1.4E2	2.7E5	1.9E3	1.8E2	2.9E5	1.6E3
	9	31.8	$\geq$ 3E136	$\geq$ 1E135	92	1.4E5	1.5E3	1.4E2	1.9E5	1.4E3
	10	21.2	$\geq$ 2E106	$\geq$ 9E104	65.8	7.9E4	1.2E3	110	1.3E5	1.2E3
	11	16.9	$\geq$ 5E81	$\geq$ 3E80	49.8	4.7E4	9.4E2	90	9.2E4	1.0E3
2	12	14.5	$\geq$ 2E61	$\geq$ 1E60	39.6	3.0E4	7.5E2	76	6.8E4	8.9E2
	15	11.2	$\geq$ 2E17	$\geq$ 2E16	24.4	9.1E3	3.7E2	50.3	3.1E4	6.2E2
	20	9.26	17	1.9	16.1	2.2E3	1.4E2	32.3	1.2E4	3.6E2
	30	7.95	7.4	.93	11.5	3.5E2	30	20.2	3.2E3	1.6E2
	50	7.18	4.1	.58	9.38	43	4.5	13.9	6.6E2	48
	100	6.70	2.4	.36	8.25	6.2	.75	10.8	94	8.8
	150	6.56	1.9	.28	7.94	3.8	.47	9.96	34	3.4
	200	6.50	1.6	.24	7.80	3.0	.38	9.59	16	1.7
	300	6.43	1.2	.19	7.65	2.3	.30	9.26	8.4	.91
	400	6.40	1.1	.17	7.59	1.9	.25	9.10	5.4	.59
	500	6.38	.94	.15	7.55	1.7	.22	9.01	4.0	.44
	1000	6.34	.65	.10	7.47	1.1	.15	8.83	2.2	.25
	∞	6.30	0	0	7.39	0	0	8.65	0	0