The Joint Economic-statistical Design of $\overline{X}$ and R Charts for Nonnormal Data

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Abstract

We consider the joint economic-statistical design of $\overline{X}$ and R control charts under the assumption that the quality measurement and the in-control time have Johnson and Weibull distributions. The Johnson distribution is general in that it can be made to fit all possible values of skewness and kurtosis. The four parameters—the sample size n, time h between successive samples, and the control factors k_1 and k_2 for the $\overline{X}$ and R charts—are determined so that the mean hourly loss-cost is minimized under constrains on the Type I and II error probabilities. We have generalized the Costa model to accommodate the Johnson and Weibull distributions. Sensitivity to nonnormality, shift, and Weibull scale parameter are considered in our analysis. Our sensitivity analysis shows that the optimal design parameters are sensitive to nonnormality. Comparisons of the fully economic and economic-statistical designs are given.

Keywords: Economic-statistical design, Johnson distribution, R chart, $\overline{X}$ chart, Weibull distribution

1 Introduction

The $\overline{X}$ and R control charts are SPC (statistical process control) tools used to monitor the production process and detect in-control to out-of-control transitions attributable to assignable causes. A production process often operates in the in-control state for a period of time. However, when an assignable cause, such as aging effects or wear, eventually occurs,

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it may shift the process mean and/or the process variance to out-of-control values. When the process is out of control, the rate of nonconformity increases and the production cost increases. The $\overline{X}$ chart is useful for monitoring a change in the process mean. The R chart is used to monitor a change, especially an increase, in the process variance. After the assignable causes are identified and verified, an out-of-control action plan is activated to eliminate them, bring the process back into a state of control, and find the underlying root cause of the problem. The $\overline{X}$ and R charts should be used simultaneously because it is essential to maintain control over both the process mean and process variability.

When $\overline{X}$ and R control charts are used jointly, four parameters—sample size n, time h between successive samples, and factors k_1 and k_2—must be determined. Since Duncan [1] first proposed the economic design of the $\overline{X}$ chart for normally distributed quality characteristics, numerous studies on the $\overline{X}$ chart, R chart, and joint design of $\overline{X}$ and R charts have been published ([2], [3], [4]). The literature on the joint design of $\overline{X}$ and R charts can be grouped into three categories: statistical, economic, and economic-statistical designs.

The statistical design chooses optimal design values under desired Type I and II error probability constraints. In this category, Saniga [5] proposed the joint statistical design for $\overline{X}$ and R control charts. Bai and Choi [6] and Chan and Cui [7] provided asymmetric control limits for $\overline{X}$ and R control charts for skewed populations. De Magalhães et al. [8] proposed a joint statistical design of adaptive $\overline{X}$ and R charts.

The economic design chooses optimal design values so that the expected hourly loss-cost is minimized. Saniga [9] first proposed an economic model for $\overline{X}$ and R control charts, assuming that the occurrence of one assignable cause precludes the occurrence of the other. In a later paper, Saniga [10] investigated the effects of alternate process models. Saniga and Montgomery [11] developed models for a process subject to a single assignable cause, assuming that its occurrence results in a simultaneous shift in the process mean and variance. Jones and Case [12] presented an economic model, based on Duncan's cost model, which assumes that the occurrence times for the two assignable causes are independently and exponentially distributed. Likewise, Costa's model [13] assumed that the two assignable causes (resulting in shifts in the mean and variance) occur independently but Costa's model is based on the Lorenzen-Vance model [14]. Costa's work employed an algorithm by Rahim [15] that determines the optimal $\overline{X}$ and R chart parameters based on these assumptions. Following Duncan's approach, Costa and Rahim [16] provided an economic design of $\overline{X}$

and R control charts under Weibull shock models for controlling normal variable quality characteristics. Like Banerjee and Rahim [17], they allowed for a non-uniform sampling interval by assuming a Weibully distributed time to failure. Ohta et al. [18] proposed an economic model for time-varying control charts in online monitoring of the mean and variance of a normally distributed quality characteristic.

The economic-statistical design chooses design parameter values that minimize the expected hourly loss-cost under constraints on the Type I and II error probabilities—or equivalently, under constraints on in-control and out-of-control average run lengths. Although the loss-cost for the economic-statistical design is higher than that of the economic design, the economic-statistical design can prevent high risks on having Type I and II errors [19]. Saniga [20] developed an economic-statistical design for Shewhart-type control charts and applied it to the joint determination of $\overline{X}$ and R chart parameters under the assumption that the quality characteristic measurement and in-control time have normal and exponential distributions. McWilliams et al. [21] presented an algorithm and FORTRAN code, based on the Lorenzen-Vance model [14], for joint determination of $\overline{X}$ and R chart parameters or $\overline{X}$ and S chart parameters, assuming that the occurrence of a single assignable cause results in a simultaneous shift in the process mean and process variance.

Most literature on the joint economic-statistical design of $\overline{X}$ and R charts makes two assumptions: (i) the quality characteristic follows a normal distribution; and (ii) when the process is in control, the time that will elapse before the occurrence of an assignable cause has an exponential distribution. In many data processes, assumption (i) may not hold, e.g., Example 2 of Chen and Schmeiser [22]. Several studies have discussed the effect of nonnormality on the control limits ([23], [24], [25]).

Several authors have published work on nonnormal quality measurements for various chart types. Lashkari and Rahim [26] consider the CUSUM chart, and Nagendra and Rai [27] and Rahim [28] consider the economic design of $\overline{X}$ charts. All of this work models the probability density function of the quality measurement X by the first four terms of the Edgeworth series, which are functions of the first four moments. Because the normal distribution is a special case of the Edgeworth series, the work of these authors may be seen as a generalization on the normality assumption. An alternate approach is employed by Burr [23], Yourstone and Zimmer [25] and Chou et al. [29], who model $\overline{X}$ as a Burr distribution [30] using the moment method. The advantage of the Burr distribution is that it has a closed-form cdf (cumulative distribution function), which simplifies computations of the Type I and II error probabilities. The disadvantage is that because the Burr distribution is right skewed, unlike the Edgeworth series, it strictly limits $\overline{X}$ to a nonnormal distribution.

For nonnormal data, another alternative (besides the Burr distribution and Edgeworth series) is to model the quality measurement having a Johnson distribution [31]. The advantage of the Johnson family (including normal, lognormal, bounded, and unbounded types) is that it covers the entire feasible part of the (β_1, β_2) plane, where β_1 denotes the squared skewness and β_2 the kurtosis. All lognormal (β_1, β_2) fall on the lognormal curve. The region above the lognormal curve consists of bounded Johnson distributions, denoted by S_B. The region below, which consists of unbounded Johnson distributions, we denote S_U. For each point (β_1, β_2), there is one corresponding Johnson distribution ([32], page 36). All (β_1, β_2) for the Edgeworth series fall on a curve below the lognormal curve; all (β_1, β_2) for the Burr distribution form a region that is above the Weibull curve in the Pearson $(\sqrt{\beta}_1, \beta_2)$ plane ([32], pages 29 and 687). The Johnson family accommodates a wide range of skewness and kurtosis.

Our sensitivity analysis, presented in Section 3 of this paper, also shows that nonnormality has a significant effect on the design parameters and hence should not be ignored. Furthermore, because the exponential distribution is memoryless, it may not accurately model the behavior of the assignable causes. In contrast, the Weibull distribution has three advantages: numerous distribution shapes (including the exponential shape), a nonconstant hazard rate, and a closed-form cdf.

The present research employs a joint economic-statistical approach. We choose parameter values that minimize the mean hourly loss-cost while maintaining reasonable Type I and Type II error probabilities. We extend the Costa cost model so that the quality characteristic measurement X and the in-control time have Johnson and Weibull distributions. In this way, we generalize on previous work in the joint design of $\overline{X}$ and R charts, which assumes the quality characteristic to be normally distributed and the in-control time to be exponentially distributed. We also avoid the disadvantages associated with these assumptions, which we have described above.

The rest of this paper is organized as follows. In Section 2, we discuss the loss-cost function and constraints under the Johnson and Weibull assumptions. In Section 3, we perform sensitivity analysis to determine the effects of nonnormality, shift, and Weibull scale parameter on the optimal parameters. Comparisons of the fully economic and economic-statistical designs are also given. Section 4 gives our conclusions. Appendix I lists key notations used in this paper. Appendix II contains a profile of the Johnson distribution family.

2 The Cost Model

Our cost model extends the work of Costa [13] by assuming that the quality characteristic measurement X has a Johnson distribution and the time until an assignable cause occurs follows a Weibull distribution. We choose Costa's model because it allows for independent occurrences of the two assignable causes of variation. The production process is assumed to start in an in-control state, where the quality measurement X has mean μ_0, standard deviation σ_0, skewness α_3 $(= \mathrm{E}[(X-\mu_0)/\sigma_0]^3)$ and kurtosis α_4 $(= \mathrm{E}[(X-\mu_0)/\sigma_0]^4)$. When assignable cause 1 occurs, the process mean μ shifts from μ_0 to $\mu_0 + \delta\sigma_0$, where $\delta \in R$ and $\delta \neq 0$. When assignable cause 2 occurs, the process standard deviation σ shifts from σ_0 to $\gamma\sigma_0$, with $\gamma > 1$. The two assignable causes occur independently. The production process is shut down during the assignable-cause search and during repair time. The elapsed time T_i before assignable cause i occurs follows the Weibull(θ, λ_i) distribution with mean $(1/\lambda_i)^{1/\theta}\Gamma(1+1/\theta)$, where θ and λ_i are the shape and scale parameters (i = 1, 2) and $\Gamma(\cdot)$ is the gamma function. Since assignable causes 1 and 2 occur independently, $T_{\min}$ $= \min(T_1, T_2)$ has a Weibull(θ, λ) distribution, where $\lambda = \lambda_1 + \lambda_2$. It follows that the duration of the in-control state has a Weibull(θ, λ) distribution.

To detect a shift in the process mean and/or process variance, a sample of n independent quality characteristic measurements $X_1, ..., X_n$ is taken at intervals of h hours. Values of the sample average $\overline{X}$ and range $R = X_{(n)} - X_{(1)}$ are recorded in the $\overline{X}$ and R charts, where $X_{(n)}$ and $X_{(1)}$ denote the largest and smallest observations in the sample. The production process is deemed out of control under the condition that (i) $\overline{X}$ falls outside the $\overline{X}$-chart control limits $\mu_0 \pm k_1\sigma_0/\sqrt{n}$; or/and (ii) the range R exceeds the R-chart upper limit $k_2\sigma_0$. The R-chart lower limit is set to zero for simplicity. When a point ($\overline{X}$ or R) falls outside the control limits, a signal is recorded in the appropriate chart, and the quality-control engineers try to locate an assignable cause. In this case, the quality-control engineers try to determine whether there is an assignable cause. If such an assignable cause is identified, then appropriate action is taken on the production process to restore the in-control state.

In an economic-statistical design, the four design parameters n, h, k_1, and k_2 are chosen so that the expected hourly loss-cost is minimized under constraints on the Type I and II error probabilities. (The values of the mean shift δ and the variance shift γ are assumed known so that the optimal values of n, h, k_1, and k_2 can be computed.) The expected hourly profit is defined as $\mathrm{E}(C)/\mathrm{E}(T)$, where $\mathrm{E}(C)$ is the expected net profit in a production cycle and $\mathrm{E}(T)$ is the expected cycle length. A production cycle starts with an in-control state, enters the out-of-control state, and ends when the assignable cause(s) are removed. Based on [13],

$$\mathrm{E}(T) = (1/\lambda)^{1/\theta}\,\Gamma(1+1/\theta) + T_X + T_R + T_{XR} + \alpha s D_0 + D_1,$$

where T_X is the expected duration of an out-of-control process mean and in-control process variance, T_R is the expected duration of an out-of-control process variance and in-control process mean, and T_{XR} is the expected duration of an out-of-control process mean and out-of-control process variance. Furthermore, α, the joint Type I error probability for the $\overline{X}$ and R control charts, can be computed as

$$\alpha = \mathrm{P}\{\ |\overline{X} - \mu_0| > k_1\sigma_0/\sqrt{n} \quad \text{or} \quad R > k_2\sigma_0 \mid \mu = \mu_0, \sigma = \sigma_0\ \}. \tag{1}$$

The number s of samples taken during the in-control state is

$$s = \sum_{i=0}^{\infty}\ i\ \mathrm{P}\{ih \le T_{\min} < (i+1)h\} = \sum_{i=1}^{\infty}\ e^{-\lambda(ih)^{\theta}}, \tag{2}$$

as shown in McWilliams [33]. The parameters D_0 and D_1 represent the expected search time for a false alarm and the expected search and process adjustment time for the occurrence of the first and/or second assignable causes.

Using Costa's logic [13] and the fact that the duration of the in-control state is Weibully distributed, we can compute T_X, T_R, and T_{XR}. Suppose that the process goes out-of-control during the mth sampling interval. Then there are three possible explanations, or cases: (i) only the process mean is out-of-control; (ii) only the process variance is out-of-control; (iii) both the process mean and the process variance are out of control.

First, consider T_X, the expected duration during a production cycle of the out-of-control mean/in-control variance state. Let U denote the actual duration, so that $T_X = \mathrm{E}[U]$. Since

U is nonzero only when assignable cause 1 occurs before assignable cause 2, we know a priori that $\mathrm{E}[U|\text{case(ii)}] = 0$. In case (i), there are two possibilities: either assignable cause 1 is detected before any shift in the variance or assignable cause 1 is detected after a shift in the variance. When the first possibility occurs, the expected duration during a production cycle of the out-of-control mean/in-control variance state equals

$$A_1 = \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} [(t+1)h - \tau_1]\ (1-p_1)^t\ p_1 \left\{F_{T_1}[(m+1)h] - F_{T_1}(mh)\right\} \left\{1 - F_{T_2}[(m+t+1)h]\right\},$$

where

$$p_1 = \mathrm{P}\left\{|\overline{X} - \mu_0| > k_1\sigma_0/\sqrt{n} \text{ or } R > k_2\sigma_0 \ |\mu = \mu_0 + \delta\sigma_0, \sigma = \sigma_0\right\} \tag{3}$$

is the joint power of the $\overline{X}$ and R control charts when only the process mean shifts,

$$\tau_i = \mathrm{E}[T_i - mh \mid mh < T_i < (m+1)h] = \frac{\int_{mh}^{(m+1)h}(t_i - mh) f_{T_i}(t_i)\ dt_i}{F_{T_i}[(m+1)h] - F_{T_i}(mh)}$$

is the mean time between the occurrence of the ith assignable cause and its last sampling time, $i = 1,\ 2$, and F_{T_i} and f_{T_i} are the cdf and pdf (probability density function) of the Weibully distributed random variable T_i.

On the other hand, if the second possibility in Case (i) occurs, the expected duration during a production cycle of the out-of-control mean/in-control variance state equals

$$A_2 = \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} [(t+1)h - \tau_1 + \tau_2](1-p_1)^{t+1}\left\{F_{T_1}[(m+1)h] - F_{T_1}(mh)\right\} \left\{F_{T_2}[(m+t+2)h] - F_{T_2}[(m+t+1)h]\right\}.$$

For Case (iii), if the mean shifts before the variance, the expected duration during a production cycle of the out-of-control mean/in-control variance state is then

$$A_3 = \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h}\int_{t_1}^{(m+1)h} (t_2 - t_1) f_{T_2}(t_2) f_{T_1}(t_1) dt_2 dt_1.$$

Therefore, the expected total time in which only the mean is out of control is

$$T_X = A_1 + A_2 + A_3. \tag{4}$$

The derivation of T_R is analogous to that of T_X. For Case (i), the expected time in which only the variance is out of control (abbreviated as ETV) is zero. For Case (ii), ETV equals B_1 when the mean does not shift before assignable cause 2 is detected and B_2, otherwise, where

$$\begin{aligned} B_1 &= \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} [(t+1)h - \tau_2]\ (1-p_2)^t\ p_2 \left\{F_{T_2}[(m+1)h] - F_{T_2}(mh)\right\} \\ &\quad \left\{1 - F_{T_1}[(m+t+1)h]\right\} \end{aligned}$$

and

$$\begin{aligned} B_2 &= \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} [(t+1)h - \tau_2 + \tau_1](1-p_2)^{t+1} \\ &\quad \left\{F_{T_2}[(m+1)h] - F_{T_2}(mh)\right\}\left\{F_{T_1}[(m+t+2)h] - F_{T_1}[(m+t+1)h]\right\}. \end{aligned}$$

Notice that

$$p_2 = \mathrm{P}\left\{|\overline{X} - \mu_0| > k_1\sigma_0/\sqrt{n} \ \text{ or } \ R > k_2\sigma_0 \ |\mu = \mu_0, \sigma = \gamma\sigma_0\right\} \tag{5}$$

is the joint power when only the process variance shifts. For Case (iii), ETV equals B_3 when the variance shifts before the mean, where

$$B_3 = \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h} \int_{t_2}^{(m+1)h} (t_1 - t_2) f_{T_1}(t_1) f_{T_2}(t_2) dt_1 dt_2.$$

Hence, the expected total time in which only the variance is out of control is

$$T_R = B_1 + B_2 + B_3. \tag{6}$$

Computation of T_{XR}, the expected time in which both the process mean and variance are out of control (abbreviated as ETMV), also occurs in three steps. For Case (i), when the variance goes out of control before assignable cause 1 is detected, ETMV equals

$$\begin{aligned} C_1 &= \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} [h/p_3 - \tau_2](1-p_1)^{t+1}\Big\{F_{T_1}[(m+1)h] - F_{T_1}(mh)\Big\} \\ &\qquad \Big\{F_{T_2}[(m+t+2)h] - F_{T_2}[(m+t+1)h]\Big\}. \end{aligned}$$

Likewise, in Case (ii), if the mean goes out of control before assignable cause 2 is detected, ETMV equals

$$\begin{aligned} C_2 &= \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} [h/p_3 - \tau_1](1-p_2)^{t+1}\Big\{F_{T_2}[(m+1)h] - F_{T_2}(mh)\Big\} \\ &\qquad \Big\{F_{T_1}[(m+t+2)h] - F_{T_1}[(m+t+1)h]\Big\}. \end{aligned}$$

In Case (iii), ETMV equals C_{3a} if the mean shifts before the variance and C_{3b}, otherwise, where

$$\begin{aligned} C_{3a} &= \sum_{m=0}^{\infty} \mathrm{E}[(m+1)h - T_2 \mid mh < T_1 < T_2 < (m+1)h]\ \mathrm{P}\{mh < T_1 < T_2 < (m+1)h\} \\ &\quad + \sum_{m=0}^{\infty}\sum_{t=1}^{\infty}(t-1)h(1-p_3)^{t+1}\ p_3\ \mathrm{P}\{mh < T_1 < T_2 < (m+1)h\} \\ &= \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h}\int_{t_1}^{(m+1)h}[(m+1)h - t_2] f_{T_1}(t_1) f_{T_2}(t_2) dt_2 dt_1 \\ &\quad + \sum_{m=0}^{\infty} h(1/p_3 - 1)\ \mathrm{P}\{mh < T_1 < T_2 < (m+1)h\} \end{aligned}$$

and

$$\begin{aligned} C_{3b} &= \sum_{m=0}^{\infty} \mathrm{E}[(m+1)h - T_1 \mid mh < T_2 < T_1 < (m+1)h]\ \mathrm{P}\{mh < T_2 < T_1 < (m+1)h\} \\ &\quad + \sum_{m=0}^{\infty}\sum_{t=1}^{\infty}(t-1)h(1-p_3)^{t+1}\ p_3\ \mathrm{P}\{mh < T_2 < T_1 < (m+1)h\} \\ &= \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h}\int_{mh}^{t_1}[(m+1)h - t_1] f_{T_1}(t_1) f_{T_2}(t_2) dt_2 dt_1 \\ &\quad + \sum_{m=0}^{\infty} h(1/p_3 - 1)\ \mathrm{P}\{mh < T_2 < T_1 < (m+1)h\}. \end{aligned}$$

Notice that

$$p_3 \;=\; \mathrm{P}\Big\{|\overline{X} - \mu_0| > k_1\sigma_0/\sqrt{n} \;\text{ or }\; R > k_2\sigma_0 \,|\mu = \mu_0 + \delta\sigma_0, \sigma = \gamma\sigma_0\Big\} \tag{7}$$

is the joint power when both the process mean and variance shift. Therefore, the total expected time in which both the mean and variance shift is

$$T_{XR} = C_1 + C_2 + C_3, \tag{8}$$

where $C_3 = C_{3a} + C_{3b}$.

Similarly, the expected net profit of the entire cycle is

$$\begin{aligned} \mathrm{E}(C) \;=\; & V_0[(1/\lambda)^{1/\theta}\Gamma(1 + 1/\theta)] + V_1 T_X + V_2 T_R + V_3 T_{XR} \\ & -(a_1 + a_2 n)[(1/\lambda)^{1/\theta}\Gamma(1 + 1/\theta) + T_X + T_R + T_{XR}]/h - a_3 - a_4\,\alpha\, s, \end{aligned}$$

where V_0, V_1, V_2, and V_3 are the profit per hour produced while the process is in control, while only the process mean is out of control, while only the process variance is out of control, and while both the mean and variance are out of control. The constants a_1 and a_2 represent the fixed and variable costs per sample; a_3 represents the expected cost of locating and eliminating assignable causes 1 and/or 2; and a_4 represents the search cost associated with each false alarm.

The hourly profits V_0, V_1, V_2, and V_3 depend on the defect rate, defined as $\mathrm{P}\{|X - \mu_0| > 3.5\sigma_0\}$. Specifically, we set $V_0 = V - 1000\ \mathrm{P}\{|X - \mu_0| > 3.5\sigma_0|\mu = \mu_0, \sigma = \sigma_0\}$, $V_1 = V - 1000\ \mathrm{P}\{|X - \mu_0| > 3.5\sigma_0|\mu = \mu_0 + \delta\sigma_0,\ \sigma = \sigma_0\}$, $V_2 = V - 1000\ \mathrm{P}\{|X - \mu_0| > 3.5\sigma_0|\mu = \mu_0, \sigma = \gamma\sigma_0\}$, and $V_3 = V - 1000\ \mathrm{P}\{|X - \mu_0| > 3.5\sigma_0|\mu = \mu_0 + \delta\sigma_0, \sigma = \gamma\sigma_0\}$. Here, V is the profit per hour when all items fall within the specification limits $\mu_0 \pm 3.5\sigma_0$. Therefore, each .001 increase in the defect rate decreases the profit per hour by one dollar. The multipliers 1000 (associated with the defect cost) and 3.5 (associated with the specification limits) were chosen arbitrarily and can be specified by the user.

Following [13], we define the expected hourly loss-cost as

$$F = V_0 - \mathrm{E}(C)/\mathrm{E}(T).$$

In the joint economic-statistical design, the design parameters n, h, k_1, and k_2 are chosen to minimize the expected hourly loss-cost associated with a production cycle under constraints on the Type I and II error probabilities α and β. Notice that the Type II error probability β equals one minus the joint power, defined in Equations (3), (5), and (7). That is, β equals $1-p_1$ if $\mu \neq \mu_0$ and $\sigma = \sigma_0$, $1-p_2$ if $\mu = \mu_0$ and $\sigma > \sigma_0$, and $1-p_3$ if $\mu \neq \mu_0$ and $\sigma > \sigma_0$. Let r_1 and r_2 denote the upper bounds for α and β, respectively. Then the design parameters n, h, k_1, and k_2 are determined by solving the optimization problem:

$$\begin{aligned} \min \quad & F \\ \text{s.t.} \quad & \alpha < r_1 \\ & \beta < r_2 \\ & n \in \{2, 3, ...\}, h > 0, k_1 > 0, k_2 > 0. \end{aligned} \tag{9}$$

Since the objective function F is not unimodal with respect to the design parameters n, h, k_1, and k_2, we cannot use optimization methods designed for unimodal functions. Therefore, we use the grid search method to determine the optimal values n^*, h^*, k_1^* and k_2^* of n, h, k_1, and k_2, as in [21]. The Type I and II error probabilities α and β can be easily computed only for special cases, e.g., normal population, because α and β depend upon the joint distribution of $\bar{X}$ and R. Hence, we compute α and β via simulation experiments.

3 Sensitivity Analysis

In this section, we conduct simulation experiments to determine the effect of the input variables on the values of the optimal design parameters. Specifically, we investigate how changes in the mean shift δ, the variance shift γ, the Weibull scale parameters λ_1 and λ_2, and the nonnormality (α_3, α_4) affect the values of the sample size n, the sampling interval h, and the factors k_1 and k_2. Based on this sensitivity analysis, we compare the performance of the economic-statistical and fully economic designs.

Tables 1 through 4 employ four distinct pairs of skewness and kurtosis values, corresponding to four distributions: normal ((α_3, α_4) = (0,3) in Table 1), bounded Johnson ((α_3, α_4) = (2, 6) in Table 2), lognormal ((α_3, α_4) = (2, 10.8634) in Table 3), and unbounded Johnson ((α_3, α_4) = (5, 100) in Table 4). Equations (1), (3), (5), and (7) show that the values of α and β, and hence n^*, h^*, k_1^*, and k_2^*, are functionally independent of the Johnson mean μ_0 and standard deviation σ_0. In the simulation experiments, we arbitrarily set $\mu_0 = 0$ and $\sigma_0 = 1$.

In each table, the mean-shift $\delta \in \{1, 2\}$, the variance-shift $\gamma \in \{1.5, 2\}$, the Weibull scale parameters $(\lambda_1, \lambda_2) \in \{(.0141, .1273), (.0707, .0707), (.1273, .0141)\}$, and the Type I and Type II error probability upper bounds $(r_1, r_2) \in \{(1, 1), (.1, .1), (.0027, .005)\}$. All together, there are 144 (=4·2·2·3·3) experimental points. For each experimental point, the optimal design parameters n^*, h^*, k_1^*, and k_2^* are calculated using the grid search method.

In each table, columns 1 to 6 show the mean shift δ, the variance shift γ, the Weibull scale parameters (λ_1, λ_2), and the upper bounds r_1 and r_2 on the Type I and II error probabilities. Columns 7 to 10 show the optimal design parameters n^*, h^*, k_1^*, and k_2^*. Columns 11 to 13 show the Type I error probability α, the Type II error probability β, and the expected hourly loss-cost F.

The other cost parameter values are as follows. The expected search time D_0 for one false alarm is .1 hours. The expected search and process adjustment time D_1, following the occurrence of assignable causes 1 and/or 2, is .3 hours. The fixed and variable sampling costs are a_1= .5 and a_2= .1. The expected cost a_3 required to locate and eliminate assignable causes 1 and/or 2 is 2. The expected search cost a_4 for each false alarm is 1. The hourly profit V when all items fall within the specification limits is 120. The Weibull shape parameter θ is .5 for both assignable causes. Notice that since $\lambda = \lambda_1 + \lambda_2 = .1414$, we have chosen both the Weibull scale parameters λ_1 and λ_2 and the Weibull shape parameter so that the mean time before the occurrence of an assignable cause is constant at 100 hours.

First, we consider the sensitivity of the optimal design parameters n^*, h^*, k_1^*, and $k_2{}^*$ to changes in the mean shift δ. Regardless of the distribution shape, when δ increases, the sample size n^* decreases, the time h^* between successive samples decreases, and the $\overline{X}$-chart factor k_1^* usually increases. These effects are the same as in Costa [13] and can be explained as follows. When δ becomes large, it is easier to detect that the process mean is out of control and hence, the sample size n^* need not be large. Usually, a decrease in n^* leads to a decrease in h^*. This is because the sampling cost for a smaller sample is lower, and hence more frequent sampling is allowed. Furthermore, since a large δ results in a lower Type II error probability β, the factor k_1^* increases, reducing the Type I error probability α

while maintaining an allowable value of β. An increase in the mean shift δ has a differing effect on the value of the R-chart factor k_2^*, depending on the distribution shape. In the normal case (Table 1), k_2^* decreases. In the bounded Johnson case (Table 2), k_2^* remains nearly constant. In the lognormal and unbounded Johnson distributions (Tables 3 and 4), k_2^* usually increases.

Second, we consider the sensitivity of the optimal design parameters n^*, h^*, k_1^*, and k_2^* to changes in the variance shift γ. For the normal population (Table 1), an increase in γ often results in lower values of n^* and h^* and higher values of k_2^*. The reason is the same as for changes in δ. For the other three populations (Tables 2 to 4), when r_1 and r_2 are .1 or bigger, the effect of changing γ on n^*, h^*, and k_2^* is similar to that for the normal population. However, when $r_1 = .0027$ and $r_2 = .005$, an increase in γ often results in higher values of n^*. Chen and Cheng [34] show that in an economic-statistical design with a small skewness, α and β with large values of k_1 and k_2 increase as the kurtosis increases. Hence, when r_1 and r_2 are small and kurtosis is high, n^* needs to be large; consequently n^* increases with γ.

Next, we consider the effect of changes in the Weibull scale parameters (λ_1, λ_2) on the values of the optimal design parameters. For each (δ, γ) pair, Tables 1 to 4 list three (λ_1, λ_2) values: one in which $\lambda_1 < \lambda_2$ (.0141, .1273), one in which $\lambda_1 = \lambda_2$ (.0707, .0707), and one in which $\lambda_1 > \lambda_2$ (.1273, .0141). (Recall that we have chosen λ_1 and λ_2 so that $\lambda = \lambda_1 + \lambda_2 = .1414$, i.e., the mean time before the occurrence of an assignable cause is constant at 100 hours.) In all four tables, as λ_1 increases relative to λ_2, n^* and k_1^* usually decrease and k_2^* usually increases. These effects are the same as in Costa [13] and can be explained as follows. As λ_1 increases, the frequency of assignable cause 1 relative to assignable cause 2 increases, and hence k_1^* decreases and k_2^* increases. (Similarly, as λ_2 increases, k_2^* decreases and k_1^* increases.) Moreover, the effect on h^* depends on δ; h^* usually increases for $\delta = 1$ and decreases for $\delta = 2$.

Finally, we consider the effect of changing distribution shape on the values of the optimal design parameters. The further (α_3, α_4) deviates from (0,3), the further n^*, h^*, k_1^*, and k_2^* stray from the normal values listed in Table 1. Furthermore, the optimal parameters are sensitive to kurtosis α_4. Tables 2 and 3 (both with skewness $\alpha_3 = 2$) shows that when r_1 and r_2 are large (e.g., $\geq .1$), n^* and k_1^* decrease as α_4 increases (while the values of h^* and k_2^* do not change monotonically). However, when r_1 and r_2 are small (e.g., $r_1 = .0027$ and

$r_2 = .005$), an increase in α_4 results in higher values of n^*. This agrees with earlier findings by Chen and Cheng [34].

Tables 1 to 4 also illustrate the relative benefits of the fully economic and economic-statistical designs. When $r_1 = r_2 = 1$, the economic-statistical design is called the fully economic design because there are no limits on the Type I and Type II error probabilities. As the upper bounds (r_1 and r_2) of α and β are relaxed, the mean hourly loss-cost decreases but the corresponding α and β increase. The tables show that the economic-statistical design should be used because the Type I and II error probabilities can be kept under .1 with only a less than 10% increase in the mean hourly loss-cost. Even when the Type I and II error probabilities are kept under .0027 and .005, respectively, the mean hourly loss-cost does not increase significantly for the normal and Johnson bounded cases as shown in Tables 1 and 2. By comparison, the fully economic design has the lowest loss-cost but the Type II error probability can be as high as .3. Therefore, the economic-statistical design is superior.

4 Conclusions

We consider the joint economic-statistical design of $\overline{X}$ and R control charts under the assumption that the quality measurement has a Johnson distribution and the time until the assignable cause occurs has a Weibull distribution. We choose the Johnson probability model because we can fit it to any desired first four moments. The control-chart design parameters n, h, k_1 and k_2 are determined so that the expected hourly loss-cost is minimized under constraints on the Type I and II error probabilities. We discuss the cost model and computations of the expected hourly loss-cost. Sensitivity analysis is performed to investigate the effects of shifts in mean and variance, Weibull scale parameter, and nonnormality on the optimal design parameters n^*, h^*, k_1^* and k_2^*. Five results follow: (i) When the mean-shift parameter δ increases, it is easier to detect the shift. Hence, the sample size n^* and the time h^* between sampling decrease. Since the sample size is smaller, the sampling cost is less, and hence, the $\overline{X}$-chart factor k_1^* increases. (ii) When r_1 and r_2 are large (e.g., $\geq .1$), an increase in γ often results in lower values of n^* and h^* and higher values of k_2^*, regardless of distribution shape. The reason is the same as in (i). On the other hand, when r_1 and r_2 are small (e.g., $r_1 = .0027$ and $r_2 = .005$) and the population kurtosis is high (e.g., > 3), n^* increases with γ. (iii) When $\lambda_1 > \lambda_2$, assignable cause 1 occurs more frequently than assignable cause 2, and hence k_1^* decreases. (iv) The values of n^*, h^*, k_1^*, and k_2^* are affected by nonnormality, especially kurtosis. Therefore, nonnormality can not be ignored. (v) The economic-statistical design is superior to the fully economic design because the Type I and II error probabilities often can be reduced to acceptable levels at only a slight increase in the loss-cost.

Appendix I: A List of Notations

The following notations are used in the paper:

n: sample size;

h: time (in hours) between successive samples;

k_1: control factor for the $\overline{X}$ chart;

k_2: control factor for the R chart;

$\overline{X}$: sample mean;

R: sample range;

μ_0: standard value for process mean;

σ_0: standard value for process standard deviation;

μ: process mean, equal to μ_0 when the process is in control;

σ: process standard deviation, equal to σ_0 when the process is in control;

α_3: process skewness;

α_4: process kurtosis;

δ: shift in the process mean (in unit of σ_0);

γ: shift in the process standard deviation (in unit of σ_0);

T_i: Elapsed time before assignable cause i occurs, $i = 1, 2$;

$T_{\min}$: minimum of T_1 and T_2;

θ: Weibull shape parameter;

λ_i: Weibull scale parameter for the random variable T_i, $i = 1, 2$;

λ: occurrence rate of the assignable cause, i.e., $\lambda = \lambda_1 + \lambda_2$;

s: number of samples taken during the in-control state;

a_1: fixed cost of sampling;

a_2: variable cost of sampling;

a_3: expected cost required to locate and eliminate assignable causes 1 and/or 2;

a_4: expected search cost for each false alarm;

D_0: expected search time for a false alarm;

D_1: expected search and process adjustment time resulting from the occurrence of assignable causes 1, 2, or both;

C: expected net profit of an entire production cycle;

T: length of a production cycle, which starts with an in-control state, goes through the out-of-control state, and ends when the assignable causes are removed;

U: time in which only the mean is out of control in the production cycle;

τ_i: expected time between the occurrence of the ith assignable cause and its last sampling time, $i = 1, 2$;

α: joint Type I error probability for the $\overline{X}$ and R charts;

p_1: joint power of the $\overline{X}$ and R charts when only the process mean shifts;

p_2: joint power of the $\overline{X}$ and R charts when only the process variance shifts;

p_3: joint power of the $\overline{X}$ and R charts when both the process mean and variance shift;

β: joint Type II error probability for the $\overline{X}$ and R charts;

r_1: upper bound for α in the economic-statistical design;

r_2: upper bound for β in the economic-statistical design;

T_X: expected time in which only the process mean is out of control;

T_R: expected time in which only the process variance is out of control;

T_{XR}: expected time in which both the process mean and variance are out of control;

V: profit per hour while all items fall within the specification limits;

V_0: profit per hour while the process is in control;

V_1: profit per hour while only the process mean is out of control;

V_2: profit per hour while only the process variance is out of control;

V_3: profit per hour while both the process mean and variance are out of control;

n^*, h^*, k_1^*, k_2^*: optimal value of the design parameters n, h, k_1, and k_2;

β_1: square of the population skewness, $\beta_1 = \alpha_3^2$;

β_2: population kurtosis, $\beta_2 = \alpha_4$.

Appendix II: The Johnson Probability Model

The Johnson distribution family, proposed by Johnson [31], includes three transformations of the standard normal distribution. Let Y and Z denote the Johnson and standard normal random variables, respectively. The three transformations are:

$$
\begin{aligned}
S_L: &\quad Z = \omega + \psi \ln(\frac{Y-\xi}{\eta}), &\quad \eta(\,Y-\xi) \geq 0,\\
S_B: &\quad Z = \omega + \psi \ln(\frac{Y-\xi}{\xi+\eta-Y}), &\quad 0 \leq Y-\xi \leq \eta,\\
S_U: &\quad Z = \omega + \psi \sinh^{-1}(\frac{Y-\xi}{\eta}), &\quad -\infty < Y < \infty.
\end{aligned}
\tag{10}
$$

The constants ξ and η are location and scale parameters, respectively; ω and ψ are the shape parameters. The second transformation, S_B, provides a bounded random variable Y; the third transformation, S_U, results in an unbounded Y. For lognormal distributions, S_L, the range is bounded below if $\eta > 0$ and bounded above if $\eta < 0$. Furthermore, the normal distribution, denoted as S_N, is one of the types of the Johnson distribution besides S_U, S_B, and S_L. We can use the numerical routines of Hill et al. [35] to find the Johnson distribution having the four desired moments mean, standard deviation, skewness, and kurtosis. To compute the Johnson cumulative probability $F(y) = \mathrm{P}\{Y \leq y\}$, we can transform y to z using Equation (10) and then let $F(y) = \Phi(z)$. For example, if Y is a lognormal distribution, $F(y) = \Phi[\omega + \psi \ln((y-\xi)/\eta)]$.

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Table 1: The optimal design values n^*, h^*, k_1^*, and k_2^* for the normal population (skewness $\alpha_3 = 0$ and kurtosis $\alpha_4 = 3$)

δ	γ	λ_1	λ_2	r_1	r_2	n^*	k_1^*	k_2^*	h^*	α	β	F
1	1.5	.0141	.1273	1	1	13	2.7	4.5	4.0	.0806	.0959	1.7581
		($\lambda_1 = .1\lambda$)		.1	.1	13	2.7	4.5	4.0	.0806	.0959	1.7581
				.0027	.005	52	3.3	6.8	4.7	.0026	.0024	2.8316
		.0707	.0707	1	1	13	2.5	4.7	4.4	.0602	.0969	1.5125
		($\lambda_1 = \lambda_2$)		.1	.1	13	2.5	4.7	4.4	.0602	.0969	1.5125
				.0027	.005	52	3.3	6.8	5.9	.0026	.0024	2.3023
		.1273	.0141	1	1	14	2.4	5.2	6.1	.0329	.1034	1.1179
		($\lambda_1 = .9\lambda$)		.1	.1	14	2.3	5.2	6.3	.0379	.0938	1.1187
				.0027	.005	50	3.2	6.9	8.6	.0024	.0029	1.5384
1	2	.0141	.1273	1	1	11	3.0	5.0	1.9	.0204	.0876	2.3443
				.1	.1	11	3.0	5.0	1.9	.0204	.0876	2.3443
				.0027	.005	40	3.2	6.8	3.1	.0023	.0046	3.1789
		.0707	.0707	1	1	11	2.8	5.0	2.4	.0228	.0797	1.9212
				.1	.1	11	2.8	5.0	2.4	.0228	.0797	1.9212
				.0027	.005	40	3.2	6.8	4.1	.0023	.0046	2.5571
		.1273	.0141	1	1	11	2.5	5.2	4.2	.0230	.0825	1.2507
				.1	.1	11	2.5	5.2	4.2	.0230	.0825	1.2507
				.0027	.005	40	3.1	6.9	6.5	.0026	.0049	1.5934
2	1.5	.0141	.1273	1	1	9	2.7	4.4	2.8	.0552	.0071	1.8699
				.1	.1	9	2.7	4.4	2.8	.0552	.0071	1.8699
				.0027	.005	23	3.3	6.3	2.3	.0026	$< 10^{-5}$	2.7176
		.0707	.0707	1	1	6	2.7	4.4	1.9	.0297	.0499	1.9028
				.1	.1	6	2.7	4.4	1.9	.0297	.0499	1.9028
				.0027	.005	22	3.2	6.4	2.4	.0025	.00002	2.6430
		.1273	.0141	1	1	5	2.7	4.9	1.6	.0117	.1019	1.7484
				.1	.1	5	2.7	4.7	1.7	.0148	.0981	1.7539
				.0027	.005	17	3.1	6.4	2.4	.0026	.0002	2.2503
2	2	.0141	.1273	1	1	8	3.0	4.7	1.6	.0227	.0265	2.3758
				.1	.1	8	3.0	4.7	1.6	.0227	.0265	2.3758
				.0027	.005	22	3.3	6.3	2.2	.0025	.0002	2.8420
		.0707	.0707	1	1	7	3.0	4.8	1.5	.0149	.0481	2.1593
				.1	.1	7	3.0	4.8	1.5	.0149	.0481	2.1593
				.0027	.005	15	3.1	6.3	1.8	.0026	.0039	2.5536
		.1273	.0141	1	1	5	2.8	5.0	1.5	.0088	.1224	1.8006
				.1	.1	6	2.9	5.1	1.6	.0079	.0851	1.8096
				.0027	.005	15	3.2	6.3	2.2	.0021	.0044	2.1673

Table 2: The optimal design values n^*, h^*, k_1^*, and k_2^* for population skewness $\alpha_3 = 2$ and kurtosis $\alpha_4 = 6$

δ	γ	λ_1	λ_2	r_1	r_2	n^*	k_1^*	k_2^*	h^*	α	β	F
1	1.5	.0141	.1273	1	1	18	2.6	4.1	2.1	.0071	.1349	2.5889
				.1	.1	19	2.7	4.1	2.3	.0131	.0966	2.5894
				.0027	.005	42	3.2	4.08	3.5	.0026	.0046	2.9674
		.0707	.0707	1	1	15	2.3	4.1	2.2	.0200	.1350	2.3787
				.1	.1	17	2.3	4.1	2.4	.0199	.0958	2.3896
				.0027	.005	42	3.2	4.08	3.6	.0026	.0046	2.8798
		.1273	.0141	1	1	12	2.0	4.1	2.3	.0365	.1990	2.1325
				.1	.1	16	2.1	4.1	2.7	.0303	.0886	2.1653
				.0027	.005	42	3.2	4.11	3.8	.0021	.0047	2.7661
1	2	.0141	.1273	1	1	12	2.7	4.1	1.7	.0178	.2147	2.5901
				.1	.1	18	2.7	4.1	2.2	.0161	.0986	2.6187
				.0027	.005	44	3.3	4.1	3.2	.0016	.0049	3.4417
		.0707	.0707	1	1	12	2.3	4.1	2.0	.0254	.2044	2.3797
				.1	.1	18	2.4	4.1	2.3	.0161	.0986	2.4604
				.0027	.005	44	3.3	4.1	3.4	.0016	.0049	3.1827
		.1273	.0141	1	1	12	2.1	4.1	2.3	.0304	.1983	2.1436
				.1	.1	19	2.4	4.1	2.7	.0130	.0981	2.2182
				.0027	.005	44	3.3	4.1	3.8	.0016	.0049	2.8710
2	1.5	.0141	.1273	1	1	13	3.0	4.1	1.6	.0000	.0304	2.6017
				.1	.1	13	3.0	4.1	1.6	.0000	.0304	2.6017
				.0027	.005	16	3.3	4.1	1.8	.0025	$< 10^{-5}$	2.6490
		.0707	.0707	1	1	8	2.6	4.1	1.3	.0008	.1377	2.4340
				.1	.1	11	2.7	4.1	2.4	.0199	.0958	2.4387
				.0027	.005	11	3.4	4.1	1.6	.0024	$< 10^{-5}$	2.5202
		.1273	.0141	1	1	6	2.4	4.1	1.3	.0034	.1774	2.1525
				.1	.1	8	2.6	4.1	1.4	.0012	.0946	2.1836
				.0027	.005	9	3.5	4.1	1.6	.0022	.0047	2.2344
2	2	.0141	.1273	1	1	12	2.9	4.1	1.6	.0068	.0145	2.6415
				.1	.1	12	2.9	4.1	1.6	.0068	.0145	2.6415
				.0027	.005	14	3.3	4.1	1.6	.0026	.0003	2.6714
		.0707	.0707	1	1	8	2.7	4.1	1.4	.0126	.0205	2.4450
				.1	.1	8	2.7	4.1	1.4	.0126	.0205	2.4450
				.0027	.005	13	3.4	4.1	1.6	.0022	.0035	2.5603
		.1273	.0141	1	1	5	2.5	4.1	1.3	.0067	.2856	2.1912
				.1	.1	8	3.3	4.1	1.4	.0044	.0980	2.2126
				.0027	.005	13	3.4	4.1	1.8	.0022	.0035	2.4322

Table 3: The optimal design values n^*, h^*, k_1^*, and k_2^* for population skewness $\alpha_3 = 2$ and kurtosis $\alpha_4 = 10.8634$

δ	γ	λ_1	λ_2	r_1	r_2	n^*	k_1^*	k_2^*	h^*	α	β	F
1	1.5	.0141	.1273	1	1	9	1.8	3.6	4.0	.2334	.1656	2.1680
				.1	.1	16	2.2	5.3	3.7	.0973	.0983	2.4035
				.0027	.005	46	3.2	14.9	1.4	.0026	.0039	8.8976
		.0707	.0707	1	1	9	1.7	4.2	4.0	.1593	.1817	1.9653
				.1	.1	16	2.2	5.3	3.8	.0973	.0983	2.0502
				.0027	.005	46	3.2	15.0	2.1	.0026	.0039	6.5387
		.1273	.0141	1	1	9	1.7	5.3	4.5	.0815	.2140	1.5195
				.1	.1	15	2.0	5.5	3.8	.0661	.0907	1.5903
				.0027	.005	46	3.2	15.0	4.1	.0026	.0039	3.0764
1	2	.0141	.1273	1	1	9	2.0	4.1	2.7	.1432	.2006	2.3616
				.1	.1	14	2.2	4.4	2.5	.0776	.0988	2.3854
				.0027	.005	63	3.2	14.9	1.9	.0026	.0047	8.1935
		.0707	.0707	1	1	9	2.0	4.8	3.0	.0904	.2586	2.1540
				.1	.1	15	2.2	4.9	1.8	.0685	.0956	2.1683
				.0027	.005	63	3.2	14.9	2.4	.0026	.0047	6.1660
		.1273	.0141	1	1	9	1.9	5.6	3.8	.0735	.2812	1.5473
				.1	.1	15	2.1	6.2	3.5	.0618	.0937	1.6778
				.0027	.005	63	3.2	14.9	5.2	.0026	.0047	3.1766
2	1.5	.0141	.1273	1	1	6	1.9	3.7	3.0	.1623	.0022	2.2670
				.1	.1	12	2.0	3.9	3.7	.0986	.0103	2.2965
				.0027	.005	40	3.2	16	1.3	.0026	$< 10^{-5}$	8.7621
		.0707	.0707	1	1	4	1.8	4.2	2.0	.0838	.0318	2.2040
				.1	.1	4	1.8	4.2	2.0	.0838	.0318	2.2040
				.0027	.005	40	3.2	16	1.7	.0026	$< 10^{-5}$	6.6093
		.1273	.0141	1	1	4	1.8	5.4	1.8	.0775	.0546	1.8368
				.1	.1	4	1.8	5.4	1.8	.0775	.0546	1.8368
				.0027	.005	40	3.2	16	2.9	.0026	$< 10^{-5}$	3.7800
2	2	.0141	.1273	1	1	6	2.3	4.4	2.1	.0968	.0419	2.4554
				.1	.1	6	2.3	4.4	2.1	.0968	.0419	2.4554
				.0027	.005	46	3.2	15	1.5	.0026	$< 10^{-5}$	7.6477
		.0707	.0707	1	1	4	2.0	4.8	1.7	.0640	.1391	2.2494
				.1	.1	6	2.2	5.3	1.8	.0597	.0587	2.2691
				.0027	.005	46	3.2	15	1.9	.0026	$< 10^{-5}$	5.9888
		.1273	.0141	1	1	4	2.0	5.7	1.7	.0327	.1936	1.8524
				.1	.1	6	2.1	6.0	1.8	.0268	.0907	1.9060
				.0027	.005	42	3.3	15	3.1	.0023	$< 10^{-5}$	3.0538

Table 4: The optimal design values n^*, h^*, k_1^*, and k_2^* for population skewness $\alpha_3 = 5$ and kurtosis $\alpha_4 = 100$

δ	γ	λ_1	λ_2	r_1	r_2	n^*	k_1^*	k_2^*	h^*	α	β	F
1	1.5	.0141	.1273	1	1	8	1.3	3.0	4.3	.2887	.0073	2.0557
				.1	.1	12	1.7	5.7	2.8	.0979	.0843	2.4011
				.0027	.005	54	3.7	32.7	1.6	.0026	.0017	10.6137
		.0707	.0707	1	1	6	1.3	3.3	4.5	.1945	.1596	1.7990
				.1	.1	11	1.7	5.7	3.5	.0991	.0914	1.9477
				.0027	.005	54	3.7	33.1	2.2	.0026	.0017	7.6036
		.1273	.0141	1	1	6	1.3	4.9	4.9	.1099	.1832	1.3263
				.1	.1	11	1.8	5.9	5.3	.0948	.0914	1.3555
				.0027	.005	54	3.7	33.1	6.1	.0026	.0017	3.1941
1	2	.0141	.1273	1	1	6	1.6	3.2	3.0	.1628	.2478	2.3901
				.1	.1	12	2.2	5.8	2.6	.0669	.0965	2.4563
				.0027	.005	66	3.7	29.9	1.4	.0026	.0046	10.8369
		.0707	.0707	1	1	6	1.5	3.6	3.2	.1453	.2553	2.0125
				.1	.1	11	1.6	6.0	2.8	.0568	.0975	2.1952
				.0027	.005	66	3.7	30.1	2.1	.0026	.0046	7.9489
		.1273	.0141	1	1	6	1.5	5.1	4.3	.0846	.2988	1.3511
				.1	.1	11	1.7	6.2	4.4	.0472	.0983	1.4216
				.0027	.005	66	3.7	30.1	5.0	.0026	.0046	3.6089
2	1.5	.0141	.1273	1	1	6	1.3	3.1	3.6	.193	.0002	2.1500
				.1	.1	5	1.6	3.7	2.4	.099	.0030	2.2900
				.0027	.005	30	4.1	23.1	0.9	.0026	$< 10^{-5}$	9.4283
		.0707	.0707	1	1	4	1.5	3.5	2.5	.1087	.0054	2.0425
				.1	.1	4	1.6	3.6	2.4	.0964	.0074	2.0486
				.0027	.005	30	4.1	23.1	1.3	.0026	$< 10^{-5}$	6.8681
		.1273	.0141	1	1	4	1.4	5.1	2.2	.0405	.0423	1.6472
				.1	.1	4	1.4	5.1	2.2	.0405	.0423	1.6472
				.0027	.005	30	4.1	23.1	2.7	.0026	$< 10^{-5}$	3.3390
2	2	.0141	.1273	1	1	6	2.2	3.4	2.8	.1320	.0171	2.4357
				.1	.1	6	2.4	3.8	2.0	.0999	.0299	2.4899
				.0027	.005	37	3.9	27.1	0.9	.0026	$< 10^{-5}$	9.7044
		.0707	.0707	1	1	4	1.6	3.5	2.2	.0998	.0372	2.1951
				.1	.1	4	1.6	3.5	2.2	.0998	.0372	2.1951
				.0027	.005	37	3.9	27.1	1.3	.0026	$< 10^{-5}$	7.2223
		.1273	.0141	1	1	4	1.5	5.2	2.0	.0843	.0323	1.6938
				.1	.1	4	1.5	5.2	2.0	.0843	.0323	1.6938
				.0027	.005	37	3.9	27.1	2.5	.0026	$< 10^{-5}$	3.6897