Asymptotic Results for Batch-Variance Methods in Simulation Output Analysis

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Abstract

We consider the batching methods for estimating the variance of the sample variance based on steady-state correlated data. Intensive research has been devoted to the problem of estimating the variance of the sample mean, but little to the sample variance when the desired performance measure is the population variance. The batch-variance estimator (for the variance of the sample variance) is a function of the batch variances, which are the sample variances of the batched data. By viewing the sample variance as a sample mean of squared terms, we show that the asymptotic results for the batch-variance and batch-mean estimators are analogous in two ways. First, both have the same convergence rates. Second, whether batch means or batch variances are employed, a single rule applies to both multipliers in the asymptotic formula. The constant multipliers are the same, and the other multipliers depend on the data properties, which are analogous for batch variances and batch means with squared terms. We prove these results analytically for the nonoverlapping batch-variance method and argue that they can be extended to cover the overlapping batch-variance method.

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1 Introduction

Simulation is a useful tool for analysis of complex systems. However, the attention devoted to model development is rarely matched by concern for appropriate analysis of the output data. In particular, when the goal to estimate a performance measure θ, such as the mean, variance, or quantile, two issues in output analysis arise: (i) The choice of the estimator $\hat{\theta}$. (ii) Assessment of the quality of $\hat{\theta}$, normally by calculating its mean square error (mse), defined as the sum of the squared bias and variance of $\hat{\theta}$. When the output data are identically and independently distributed (i.i.d.), usually both of these tasks are straightforward. In manufacturing systems, however, the simulation output data are usually correlated, for examples, waiting times in queue, cycle times, etc. Ignoring this dependence may underestimate the mse and hence overestimate the goodness of the point estimator.

This research considers the problem of estimating the variance of the sample variance, a common estimator for the population variance, based on simulation output data that are identically distributed but correlated to each other. Let $\{X_1, X_2, ..., X_n\}$ denote a set of simulation output data (of size n), which are stationary with population mean μ and population variance σ^2. Here we assume that the desired performance measure is $\theta = \sigma^2$. A typical estimator for $\theta = \sigma^2$ is $\hat{\theta} = S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. If $\{X_1, X_2, ..., X_n\}$ are dependent but covariance stationary, S^2 is a consistent estimator of σ^2 with $\mathrm{E}(S^2) = n(n-1)^{-1}[\sigma^2 - \mathrm{Var}(\bar{X})] = n(n-1)^{-1}\sigma^2\{1 - n^{-1}[1 + 2\sum_{h=1}^{n-1}(1 - n^{-1}h)\rho_h]\}$, where ρ_h is the lag-h autocorrelation of X (David, 1985). Moreover, Priestly (1981, p. 327) gives a large-sample approximation for the variance: $\mathrm{Var}(S^2) \approx 2\sigma^4 n^{-1} \sum_{h=-\infty}^{\infty} \rho_h^2$ for n large. For the i.i.d. special case, S^2 would be unbiased with variance $\mathrm{Var}(S^2) = \sigma^4 n^{-1}[\alpha_4 - (n-3)/(n-1)]$, where α_4 is the kurtosis of X (Wilks, 1962, p. 200). Our research problem is to estimate $\mathrm{Var}(S^2)$, for purposes such as constructing confidence intervals and knowing the statistical properties of the estimator $\hat{\text{Var}}(S^2)$.

The batching method is useful for estimating the variance of $\hat{\theta}$ for correlated data and it requires only a single long run in the simulation experiment (see Section 2). In this method, the observations are divided into b batches, and an estimator $\hat{\theta}_i$, called a batch statistic, is computed for each batch. The estimator $\hat{\text{Var}}(\hat{\theta})$ of $\text{Var}(\hat{\theta})$ then becomes a function of $\hat{\theta}_1$,..., $\hat{\theta}_b$. If the b batches do not overlap, the method is called nonoverlapping batching; otherwise, it is called overlapping batching. Unlike the observations obtained from b independent simulation short runs, the b batch statistics are correlated. Hence, analysis of the statistical properties of $\hat{\text{Var}}(\hat{\theta})$ is more complicated. A single long run is more efficient than multiple short runs because the initial biased data need to be thrown away only once Throughout this paper, we assume that the collected data $\{X_1, ..., X_n\}$ have no initial bias. Furthermore, the data are assumed to be from a stationary process.

Intensive research has been devoted to the batch-means method for $\theta = \mu$, but little for $\theta = \sigma^2$ (probably because mean is a more common performance measure than variance). Ceylan and Schmeiser (1994) provide conjectures on the consistency of OBV (overlapping-batch-variance) estimators of $\text{Var}(S^2)$ by viewing the batch variance as a batch mean. We implement their ideas to show analytically the consistency of NBV (nonoverlapping-batch-variance) estimators for data from a linear process. These results can be extended to OBV estimators.

The rest of this paper is organized as follows. In Section 2, we review the literature on batching methods and properties of linear processes. In Section 3, we show the asymptotic bias and variance results of the NBV estimator for linear processes. Here, S^2 is redefined to have denominator n (rather than $n-1$) so that S^2 is a sample mean of the squared terms $\{(X_i - \bar{X})^2, i = 1, ..., n\}$. Since the variance of S^2 with denominator $n-1$ is proportional to the variance with denominator n, one can be determined from the other. In Section 4, we conclude our results.

2 Batching methods

2.1 Batch Means

To estimate the population mean μ based on the stationary steady-state data $\{X_1, \ldots, X_n\}$ generated from a stochastic simulation experiment, we can use the sample mean $\hat{\theta} = \bar{X} = \sum_{i=1}^{n} X_i/n$ as an unbiased estimator for $\theta = \mu$. The method of nonoverlapping batch means (NBMs), proposed by Conway (1963), is a classical and conceptually straightforward method for estimating the variance of the sample mean $\bar{X}$. NBM divides the sequence of observations into b adjacent and nonoverlapping batches, each of size m (assuming $n = bm$ for simplicity). The average of data, called the batch mean, is computed for each batch. The grand average $\bar{X}$ is the average of the b batch means. The NBM estimator of $\mathrm{Var}(\bar{X})$ is therefore defined as

$$\widehat{V}^{(NBM)} = \frac{\sum_{j=1}^{b} (\bar{X}_j - \bar{X})^2}{b(b-1)} = \frac{m^2}{n(n-m)} \sum_{j=1}^{n/m} (\bar{X}_j - \bar{X})^2, \tag{1}$$

where $\bar{X}_j = \sum_{i=(j-1)m+1}^{jm} X_i/m$ is the j^{th} nonoverlapping batch mean. The NBM method seeks to obtain large batches (i.e., m is large) so that the batch means are approximately independently and normally distributed, based on the central limit theorem. The tradeoff, however, is that the number of batches will then be small to cause large variation in $\widehat{V}^{(NBM)}$. To balance this tradeoff, Schmeiser (1982) suggests using ten to thirty batches.

The method of overlapping batch means (OBMs) is introduced in Meketon (1980) and Meketon and Schmeiser (1984). OBM is similar to NBM, except that it divides n observations into $n-m+1$ overlapping batches, each of size m. The OBM estimator of $\mathrm{Var}(\bar{X})$ is defined as

$$\widehat{V}^{(OBM)} = \frac{m}{(n-m)(n-m+1)} \sum_{j=1}^{n-m+1} (\bar{X}_j - \bar{X})^2, \tag{2}$$

where $\bar{X}_j = \sum_{i=j}^{j+m-1} X_i/m$ is the j^{th} overlapping batch mean. OBM utilizes observations more efficiently, as shown in Equation (4). As for the computational demand, Goldsman and Schmeiser (1997) show that the computation time and storage requirements for both NBM and OBM are $O(n)$ and $O(1)$ for any fixed value of m. Song (1988) and Pedrosa (1994) compute the variance of $\widehat{V}^{(OBM)}$ for i.i.d. data and for linear processes, respectively. Song and Schmeiser (1993) rewrite NBM and OBM variance estimators in quadratic form for algebraic and geometric analysis of their properties.

Meketon (1980), Meketon and Schmeiser (1984), Goldsman and Meketon (1986), and Song and Schmeiser (1995) discuss the asymptotic results of batch-means estimators for covariance stationary data. Let $\gamma_0 = \sum_{h=-\infty}^{\infty} \rho_h = 1 + 2\sum_{h=1}^{\infty} \rho_h$ and $\gamma_1 = \sum_{h=-\infty}^{\infty} |h|\rho_h$ denote the sum and weighted sum of autocorrelations, respectively, where $\rho_h = \mathrm{Corr}(X_i, X_{i+h})$ is the lag-h autocorrelation. If $\sigma^2 > 0$ and both γ_1 and the fourth population moment exist and are finite,

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} nm \,\mathrm{Bias}[\widehat{V}^{(NBM)}] = -\gamma_1\sigma^2 \quad \text{and} \quad \lim_{\substack{m\to\infty \\ n/m\to\infty}} \frac{n^3}{m}\mathrm{Var}[\widehat{V}^{(NBM)}] = 2(\gamma_0\sigma^2)^2. \tag{3}$$

Furthermore, comparing the asymptotic results for NBM and OBM, we have

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} \frac{\mathrm{Bias}[\widehat{V}^{(OBM)}]}{\mathrm{Bias}[\widehat{V}^{(NBM)}]} = 1 \quad \text{and} \quad \lim_{\substack{m\to\infty \\ n/m\to\infty}} \frac{\mathrm{Var}[\widehat{V}^{(OBM)}]}{\mathrm{Var}[\widehat{V}^{(NBM)}]} = \frac{2}{3}. \tag{4}$$

Combining Equations (3) and (4), we can write

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} nm \,\mathrm{Bias}[\widehat{V}^{(Type)}] = -c_b\gamma_1\sigma^2 \quad \text{and} \quad \lim_{\substack{m\to\infty \\ n/m\to\infty}} \frac{n^3}{m}\mathrm{Var}[\widehat{V}^{(Type)}] = c_v(\gamma_0\sigma^2)^2, \tag{5}$$

where

$$c_b = \begin{cases} 1 & \text{if } Type = NBM \\ 1 & \text{if } Type = OBM \end{cases}, \qquad c_v = \begin{cases} 2 & \text{if } Type = NBM \\ 4/3 & \text{if } Type = OBM \end{cases}. \tag{6}$$

Using Equation (5), we can derive the mse optimal batch size that minimizes the asymptotic mse for NBM and OBM: $m^* = n^{1/3}(\gamma_1/\gamma_0)^{2/3} + 1$ for NBM and $m^* =$

$(3n/2)^{1/3}(\gamma_1/\gamma_0)^{2/3}+1$ for OBM. Song (1996), Pedrosa (1994), and Yeh[b] (2002) propose procedures to estimate this asymptotically optimal batch size, which depends on unknown values of γ_0 and γ_1.

2.2 Batch Variances

We use simulation to estimate many types of performance measures, with variance σ^2 (or standard deviation) being the second most common after means. Ideally we would have standard errors for every point estimator. The batching method is an easy-to-implement method for estimating the standard error for a general point estimator $\hat{\theta}$, not only $\hat{\theta} = \bar{X}$ (Schmeiser, Avramidis & Hashem, 1990). Ideally the estimator of the standard error would have good statistical properties, which means that we need good batch sizes.

Following Ceylan (1994), we redefine the point estimator S^2 of variance σ^2 as

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}. \tag{7}$$

By changing the denominator from $n-1$ to n, we view the point estimator S^2 as the sample average of the squared terms $(X_i - \bar{X})^2$, $i = 1, ..., n$. For the rest of this paper, the S^2 refers to the estimator in Equation (7). The method of batch variances arranges the n observations into b batches with batch size m and computes the sample variance, called the batch variance, for each batch. The nonoverlapping batch variance (NBV) and overlapping batch variance (OBV) are defined as $S_j^2 = \sum_{i=(j-1)m+1}^{jm}(X_i - \bar{X})^2/m$ and $S_j^2 = \sum_{i=j}^{j+m-1}(X_i - \bar{X})^2/m$, respectively. Based on the batching-method logic, the NBV and OBV estimators of $\mathrm{Var}(S^2)$ are therefore

$$\widehat{V}^{(NBV)} = \frac{m^2}{n(n-m)}\sum_{j=1}^{n/m}(S_j^2 - S^2)^2 \tag{8}$$

and

$$\widehat{V}^{(OBV)} = \frac{m}{(n-m)(n-m+1)} \sum_{j=1}^{n-m+1} (S_j^2 - S^2)^2, \tag{9}$$

respectively. Figure 1 shows how NBVs are computed. Like batch means, the batch-variance estimator requires $O(n)$ computation and $O(1)$ storage.

HERE IS FIGURE 1

Using these definitions of S^2 and S_j^2, NBV and OBV estimators, respectively, are algebraically equivalent to NBM and OBM estimators for the squared terms $(X_i - \bar{X})^2, i = 1, \cdots, n$. Ceylan and Schmeiser (1994) conjecture that NBV and OBV estimators have asymptotic results analogous to the NBM and OBM results in Equation (5) because they view the data process using the squared terms. Let $\widetilde{\rho}_h$, $\widetilde{\gamma}_0$, and $\widetilde{\gamma}_1$ denote the squared term counterparts of ρ_h, γ_0, and γ_1, respectively, i.e., $\widetilde{\rho}_h = \mathrm{Corr}[(X_i - \mu)^2, (X_{i+h} - \mu)^2]$, $\widetilde{\gamma}_0 = \sum_{h=-\infty}^{\infty} \widetilde{\rho}_h$, and $\widetilde{\gamma}_1 = \sum_{h=-\infty}^{\infty} |h| \widetilde{\rho}_h$. Then Ceylan and Schmeiser's conjecture is as follows: if $(\alpha_4 - 1)\sigma^4 > 0$, $\widetilde{\gamma}_1 < \infty$, and $\mathrm{E}(X^8) < \infty$,

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} nm \ \mathrm{Bias}[\widehat{V}^{(Type)}] = -c_b \widetilde{\gamma}_1 [(\alpha_4 - 1)\sigma^4] \ ,$$

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} n^3 m^{-1} \ \mathrm{Var}[\widehat{V}^{(Type)}] = c_v \left[\widetilde{\gamma}_0 (\alpha_4 - 1)\sigma^4\right]^2 \ , \tag{10}$$

where c_b and c_v are as defined in Equation (6). Notice that $\mathrm{Var}(X_i-\mu)^2 = (\alpha_4-1)\sigma^4$ is the analogous value of $\mathrm{Var}(X_i)$ for batch means. Ceylan (1994) empirically studies the normal AR(1) data to support this conjecture for the OBV estimator in the case of dependent data. We prove in Section 3 that this conjecture holds for NBV estimators with a linear data process (Section 2.3).

Ceylan (1994) also investigates two other variations of the overlapping batch variances, in which the grand average $\bar{X}$ is replaced by (i) the batch mean $\bar{X}_j$ and (ii) the true mean μ. Her work shows that the true mean is the best replacement, but when it is unknown (most cases), the grand average is a better choice than the batch mean, because it converges to μ more rapidly.

2.3 Linear Processes

Simulation output data are usually correlated and time dependent, for example, waiting times for parts in a queue. Here, we review linear processes of time series data in general. Several time-series models, such as autoregressive and moving average, are special cases of the linear process.

Random data $\{X_i, i = 1, 2, ...\}$ are said to follow a linear process if each component can be expressed in the form

$$X_i = \sum_{h=-\infty}^{\infty} \alpha_h \varepsilon_{i-h}, \tag{11}$$

where the α_h's are real weights and $\{\varepsilon_i\}$ is a sequence of i.i.d. random variables with mean μ_ε, variance σ_ε^2, and kurtosis $\alpha_{4,\varepsilon}$. Without loss of generality, we may assume $\mu_\varepsilon = 0$ and hence $\mathrm{E}(X_i) = 0$.

Under mild conditions, the linear process is stationary up to order four. That is, $\mathrm{E}(X_i X_{i+r} X_{i+s} X_{i+t})$ depends on r, s, and t, but not i, for any i, r, s, t. Priestley (1981) shows that if $\sum_{j=-\infty}^{\infty} \alpha_j^2 < \infty$, the linear process is covariance stationary (i.e., stationary up to order 2) with lag-h autocovariance

$$R_h = \mathrm{E}(X_i X_{i+h}) = \sigma_\varepsilon^2 \sum_{j=-\infty}^{\infty} \alpha_j \alpha_{j+h}. \tag{12}$$

Furthermore, if the α_j's are absolutely summable (i.e., $\sum_{j=-\infty}^{\infty} |\alpha_j| < \infty$), then the linear process is stationary up to order four with

$$\begin{aligned} \mathrm{E}[X_i X_{i+r} X_{i+s} X_{i+t}] &= \mathrm{E}[X_i X_{i+r}]\mathrm{E}[X_{i+s} X_{i+t}] + \mathrm{E}[X_i X_{i+s}]\mathrm{E}[X_{i+r} X_{i+t}] \\ &\quad + \mathrm{E}[X_i X_{i+t}]\mathrm{E}[X_{i+r} X_{i+s}] + k_4(r,s,t), \end{aligned} \tag{13}$$

where

$$k_4(r,s,t) = (\alpha_{4,\varepsilon} - 3)\sigma_\varepsilon^4 \sum_{h=-\infty}^{\infty} \alpha_h \alpha_{h+r} \alpha_{h+s} \alpha_{h+t} \tag{14}$$

is the fourth joint cumulant of the joint distribution of X_i, X_{i+r}, X_{i+s}, and X_{i+t} (Rosenblatt, 1985, p. 35, Priestley, 1981, p. 325, and Fuller, 1996, p. 315). Combining Equations (13) and (14), we have

$$\begin{aligned} \mathrm{E}[X_i X_{i+r} X_{i+s} X_{i+t}] &= (\alpha_{4,\varepsilon} - 3)\sigma_\varepsilon^4 \sum_{h=-\infty}^{\infty} \alpha_h \alpha_{h+r} \alpha_{h+s} \alpha_{h+t} \\ &\quad + R_r R_{t-s} + R_s R_{t-r} + R_t R_{s-r}. \end{aligned} \tag{15}$$

The definition of cumulant is based on Kendall and Stuart (1977, p. 86) and Bickel and Doksum (2001, p. 460). In Section 3, we discuss statistical properties of the NBV estimator for simulation output data from a linear process.

Autoregressive and moving-average processes are special cases of the linear process. For example, the first-order autoregressive process

$$X_i = \phi X_{i-1} + \varepsilon_i, \qquad |\phi| < 1,$$

with zero mean can be expressed as a linear process with weights $\alpha_h = \phi^{|h|}$ for $h \leq i$

and 0, otherwise. The first-order moving-average process with the form

$$X_i = \delta\varepsilon_{i-1} + \varepsilon_i, \quad |\delta| < \infty,$$

is also a linear process with weights $\alpha_0 = 1$, $\alpha_1 = \delta$, and $\alpha_h = 0$ for $h > 1$. Both processes satisfy the absolutely summable condition: $\sum_{j=-\infty}^{\infty} |\alpha_j| < \infty$. Furthermore, if the white noise ε_i follows a normal distribution, both processes are stationary Gaussian up to any finite order.

3 Statistical properties of NBV estimators

In simulation output analysis, the statistical properties of batch-mean estimators have been studied extensively, but little attention has been devoted to batch variances. Here, we analyze here the asymptotic properties of NBV estimators for simulation output data from a linear process (Section 2.3). We show that the asymptotic results for NBV estimators agree with those for NBM estimators. Specifically, we show that

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} nm \text{ Bias}[\widehat{V}^{(NBV)}] = -\tilde{\gamma}_1[(\alpha_4 - 1)\sigma^4] , \tag{16}$$

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} n^3 m^{-1}\text{Var}[\widehat{V}^{(NBV)}] = 2[\ \tilde{\gamma}_0(\alpha_4 - 1)\sigma^4\ ]^2. \tag{17}$$

Both NBV and NBM have the same convergence rate and the same values of c_b and c_v (Equation 5). The other terms in the asymptotic-result formulas depend on the data process. Recall that the sample variance $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$ is the sample average of the quadratic data $\{(X_i - \bar{X})^2\}$. Therefore, the NBV analogous values of the NBM population variance σ^2 is $\text{Var}[(X_i - \mu)^2] = (\alpha_4 - 1)\sigma^4$ (the limiting value of $\text{Var}[X_i - \bar{X}]^2$ as $n \to \infty$). The NBV analogous values of γ_0 and γ_1 for NBM are $\widetilde{\gamma}_0 = \sum_{h=-\infty}^{\infty} \widetilde{\rho}(h)$ and $\widetilde{\gamma}_1 = \sum_{h=-\infty}^{\infty} |h|\widetilde{\rho}(h)$ where $\widetilde{\rho}(h)$ is the limiting lag-h autocorrelation of the quadratic data $\{(X_i - \bar{X})^2, i = 1, ..., n\}$. Notice that Equations (16) and (17) imply that $\widehat{V}^{(NBV)}$ is an mse-consistent estimator of $\text{Var}[S^2]$.

First, consider the special case of i.i.d. data $X_1, ..., X_n$. Result 1 shows that

Equations (16) and (17) are valid for i.i.d. data.

Result 1 *Suppose that $X_1, ..., X_n$ are i.i.d. with $(\alpha_4 - 1)\sigma^4 < \infty$. Then,*

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} nm \text{ Bias}[\widehat{V}^{(NBV)}] = 0 \,, \tag{18}$$

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} n^3 m^{-1} \text{Var}[\widehat{V}^{(NBV)}] = 2\left[(\alpha_4 - 1)\sigma^4\right]^2 . \tag{19}$$

Equations (18) and (19) match Equations (16) and (17), respectively, because for i.i.d. data, $\tilde{\gamma}_1 = 0$ and $\tilde{\gamma}_0 = 1$. The proof is straightforward.

We now discuss the asymptotic properties of NBV estimators for the linear process. Result 2 calculates the asymptotic bias of NBV estimators for the linear process and Result 3 calculates the asymptotic variance. Through out this section, we assume that the number of observations is a multiple of the batch size, i.e., $n = bm$.

Result 2 *Suppose that the observations $\{X_i, i = 1, ..., n\}$ are from a linear process whose weights satisfy $\sum_{h=-\infty}^{\infty} |h\alpha_h| < \infty$. Then, for n large,*

$$\begin{aligned} & nm \text{ Bias}[\widehat{V}^{(NBV)}] \\ \approx \; & \frac{-2(\alpha_{4,\varepsilon} - 3)\sigma_\varepsilon^4}{n-m}\left[nm \sum_{h=m}^{n-1} \sum_{j=-\infty}^{\infty} \alpha_j^2 \alpha_{j+h}^2 + n \sum_{h=1}^{m-1} \sum_{j=-\infty}^{\infty} h\alpha_j^2 \alpha_{j+h}^2 - m \sum_{h=1}^{n-1} \sum_{j=-\infty}^{\infty} h\alpha_j^2 \alpha_{j+h}^2\right] \\ & - \frac{4}{n-m}\left[nm \sum_{h=m}^{n-1} R_h^2 + n \sum_{h=1}^{m-1} hR_h^2 - m \sum_{h=1}^{n-1} hR_h^2\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\substack{m\to\infty \\ n/m\to\infty}} nm \text{ Bias}\left[\widehat{V}^{(NBV)}\right] &= -2(\alpha_{4,\varepsilon} - 3)\sigma_\varepsilon^4 \sum_{h=1}^{\infty} \sum_{j=-\infty}^{\infty} h\alpha_j^2 \alpha_{j+h}^2 - 4 \sum_{h=1}^{\infty} hR_h^2 \\ &= -\widetilde{\gamma}_1[(\alpha_4 - 1)\sigma^4] \,. \end{aligned}$$

Proof:

Assume, without loss of generality, that $\mu = 0$. By the definition of NBV in

Equation (8),

$$\widehat{V}^{(NBV)} = \frac{1}{b(b-1)}\sum_{j=1}^{b}(S_j^2 - S^2)^2 = \frac{1}{b(b-1)}\Big[\sum_{j=1}^{b} S_j^4 - bS^4\Big] .$$

Now consider the batch variance S_1^2:

$$\begin{aligned} S_1^2 &= m^{-1}\sum_{i=1}^{m}(X_i - \bar{X})^2 = m^{-1}\sum_{i=1}^{m}\Big[(X_i-\mu) - (\bar{X}-\mu)\Big]^2 \\ &= m^{-1}\sum_{i=1}^{m}(X_i-\mu)^2 - (\bar{X}-\mu)^2. \end{aligned}$$

By the law of large numbers, random variable $(\bar{X}-\mu)$, as well as $(\bar{X}-\mu)^2$, converges to zero in probability, as n goes to infinity. Therefore, the first term dominates. Define $S_j^2(\mu) = \sum_{i=(j-1)m+1}^{jm}(X_i-\mu)^2/m$ and $S^2(\mu) = \sum_{i=1}^{n}(X_i-\mu)^2/n$. As n goes to infinity, $S_j^2 - S_j^2(\mu)$ and $S^2 - S^2(\mu)$ converge to zero in probability. Hence, in the rest of the proof, we approximate S_j^2 and S^2 by $S_j^2(\mu)$ and $S^2(\mu)$, respectively.

Since $\sum_{h=-\infty}^{\infty}|h\alpha_h| < \infty$, we have $\sum_{h=-\infty}^{\infty}|\alpha_h| < \infty$ and hence the linear process is stationary up to order ≥ 4. This implies that $\mathrm{E}[S_j^4]$ is the same for all j. Hence,

$$\begin{aligned} \mathrm{E}[\widehat{V}^{(NBV)}] &= \frac{\mathrm{E}[S_1^4] - \mathrm{E}[S^4]}{b-1} \\ &\approx \frac{\mathrm{E}[S_1^4(\mu)] - \mathrm{E}[S^4(\mu)]}{b-1} = \frac{m}{n-m}\Big\{\mathrm{E}[S_1^4(\mu)] - \mathrm{E}[S^4(\mu)]\Big\} . \end{aligned}$$

Therefore,

$$\begin{aligned} &nm\mathrm{Bias}[\widehat{V}^{(NBV)}] = nm\Big\{\mathrm{E}(\widehat{V}^{(NBV)}) - \mathrm{Var}(S^2)\Big\} \\ &= nm\Big\{\frac{m}{n-m}\mathrm{E}(S_1^4) - \frac{n}{n-m}\mathrm{E}(S^4) + [\mathrm{E}(S^2)]^2\Big\} \\ &\approx \frac{nm^2}{n-m}\mathrm{E}[S_1^4(\mu)] - \frac{n^2m}{n-m}\mathrm{E}[S^4(\mu)] + nm\left\{\mathrm{E}[S^2(\mu)]\right\}^2 \\ &= \frac{nm^2}{n-m}\cdot\frac{1}{m^2}\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{s=1}^{m}\sum_{t=1}^{m}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+s-t}^2 + m^2R_0^2 + 2\sum_{s=1}^{m}\sum_{t=1}^{m}R_{s-t}^2\right] \end{aligned}$$

$$-\frac{n^2 m}{n-m}\cdot\frac{1}{n^2}\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+s-t}^2+n^2R_0^2+2\sum_{s=1}^{n}\sum_{t=1}^{n}R_{s-t}^2\right]$$
$$+nmR_0^2$$
$$=\frac{n}{n-m}\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{s=1}^{m}\sum_{t=1}^{m}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+s-t}^2+2\sum_{s=1}^{m}\sum_{t=1}^{m}R_{s-t}^2\right]$$
$$-\frac{m}{n-m}\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+s-t}^2+2\sum_{s=1}^{n}\sum_{t=1}^{n}R_{s-t}^2\right].$$

Let $h=s-t$. Then

$$\sum_{s=1}^{m}\sum_{t=1}^{m}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+s-t}^2 = m\sum_{j=-\infty}^{\infty}\alpha_j^4+2\sum_{h=1}^{m-1}\sum_{j=-\infty}^{\infty}(m-h)\alpha_j^2\alpha_{j+h}^2 .$$

Similarly,

$$\sum_{s=1}^{m}\sum_{t=1}^{m}R_{s-t}^2=mR_0^2+2\sum_{h=1}^{m-1}(m-h)R_h^2.$$

Therefore,

$$\begin{aligned}
&nm\mathrm{Bias}[\widehat{V}^{(NBV)}]\\
\approx\ &\frac{n}{n-m}\Bigg\{(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\left[m\sum_{j=-\infty}^{\infty}\alpha_j^4+2\sum_{h=1}^{m-1}\sum_{j=-\infty}^{\infty}(m-h)\alpha_j^2\alpha_{j+h}^2\right]\\
&\qquad\qquad+2\left[mR_0^2+2\sum_{h=1}^{m-1}(m-h)R_h^2\right]\Bigg\}\\
&-\frac{m}{n-m}\Bigg\{(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\left[n\sum_{j=-\infty}^{\infty}\alpha_j^4+2\sum_{h=1}^{n-1}\sum_{j=-\infty}^{\infty}(n-h)\alpha_j^2\alpha_{j+h}^2\right]\\
&\qquad\qquad+2\left[nR_0^2+2\sum_{h=1}^{n-1}(n-h)R_h^2\right]\Bigg\}\\
=\ &\frac{2n}{n-m}\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{h=1}^{m-1}\sum_{j=-\infty}^{\infty}(m-h)\alpha_j^2\alpha_{j+h}^2+2\sum_{h=1}^{m-1}(m-h)R_h^2\right]\\
&-\frac{2m}{n-m}\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{h=1}^{n-1}\sum_{j=-\infty}^{\infty}(n-h)\alpha_j^2\alpha_{j+h}^2+2\sum_{h=1}^{n-1}(n-h)R_h^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4}{n-m}\Bigg[nm\sum_{h=1}^{m-1}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+h}^2 - nm\sum_{h=1}^{n-1}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+h}^2 \\
&\qquad -n\sum_{h=1}^{m-1}\sum_{j=-\infty}^{\infty}h\alpha_j^2\alpha_{j+h}^2 + m\sum_{h=1}^{n-1}\sum_{j=-\infty}^{\infty}h\alpha_j^2\alpha_{j+h}^2\Bigg] \\
&\quad + \frac{4}{n-m}\Bigg[nm\sum_{h=1}^{m-1}R_h^2 - nm\sum_{h=1}^{n-1}R_h^2 - n\sum_{h=1}^{m-1}hR_h^2 + m\sum_{h=1}^{n-1}hR_h^2\Bigg] \\
&= \frac{-2(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4}{n-m}\Bigg[nm\sum_{h=m}^{n-1}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+h}^2 + n\sum_{h=1}^{m-1}\sum_{j=-\infty}^{\infty}h\alpha_j^2\alpha_{j+h}^2 \\
&\quad -m\sum_{h=1}^{n-1}\sum_{j=-\infty}^{\infty}h\alpha_j^2\alpha_{j+h}^2\Bigg] - \frac{4}{n-m}\Bigg[nm\sum_{h=m}^{n-1}R_h^2 + n\sum_{h=1}^{m-1}hR_h^2 - m\sum_{h=1}^{n-1}hR_h^2\Bigg].
\end{aligned}
$$

Notice that under the condition $\sum_{h=-\infty}^{\infty}|h\alpha_h| < \infty$, α_h and R_h are $O(|h|^{-2-\delta})$ for $\delta > 0$, implying that $\sum_{h=1}^{\infty}\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+h}^2$, $\sum_{h=1}^{\infty}\sum_{j=-\infty}^{\infty}h\alpha_j^2\alpha_{j+h}^2$, $\sum_{h=1}^{\infty}R_h^2$ and $\sum_{h=1}^{\infty}hR_h^2$ are all finite. Therefore, as $m\to\infty$ and simultaneously $n/m\to\infty$,

$$
\begin{aligned}
\lim_{\substack{m\to\infty\\ n/m\to\infty}} nm\ \mathrm{Bias}[\widehat{V}^{(NBV)}] &= -2(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{h=1}^{\infty}\sum_{j=-\infty}^{\infty}h\alpha_j^2\alpha_{j+h}^2 - 4\sum_{h=1}^{\infty}hR_h^2 \\
&= -2\sum_{h=1}^{\infty}h\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+h}^2 + 2R_h^2\right] \\
&= -2\sum_{h=1}^{\infty}h\left\{(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4\sum_{j=-\infty}^{\infty}\alpha_j^2\alpha_{j+h}^2 + 2R_h^2 + R_0^2 - R_0^2\right\} \\
&= -2\sum_{h=1}^{\infty}h\left\{\ \mathrm{E}[X_i^2X_{i+h}^2] - \mathrm{E}[X_i^2]\mathrm{E}[X_{i+h}^2]\ \right\} = -2\sum_{h=1}^{\infty}h\ \mathrm{Cov}(X_i^2, X_{i+h}^2) \\
&= -2\sum_{h=1}^{\infty}h\ \widetilde{\rho}(h)\ \mathrm{Var}(X_i^2) = -2(\alpha_4-1)\sigma^4\sum_{h=1}^{\infty}h\ \widetilde{\rho}(h) = -\widetilde{\gamma}_1(\alpha_4-1)\sigma^4. \quad \blacksquare
\end{aligned}
$$

Result 2 verifies that Equation (16) remains valid for dependent data from a linear process. Result 2 also indicates that NBV is asymptotically unbiased with order nm.

Having established conditions sufficient to ensure that the NBV estimator is asymptotically unbiased, we now turn our attention to convergence behavior for the variance of the NBV estimator. Result 3 follows from the asymptotic independence and normality of batched variances. It shows that the NBV variance goes to zero with rate n^3/m (the same convergence rate as NBM) and conforms to Equation (17).

Result 3 *Suppose that the observations $\{X_i, i = 1, ..., n\}$ are from a linear process whose weights satisfy $\sum_{h=-\infty}^{\infty} |h\alpha_h| < \infty$. Then*

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} \frac{n^3}{m}\mathrm{Var}[\widehat{V}^{(NBV)}] = 2\left[(\alpha_{4,\varepsilon}-3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2\right]^2 = 2\left[\widetilde{\gamma}_0(\alpha_4-1)\sigma^4\right]^2.$$

Proof:

Recall that

$$\widehat{V}^{(NBV)} = \frac{m^2}{n(n-m)}\sum_{j=1}^{n/m}(S_j^2 - S^2)^2 .$$

Under condition $\sum_{h=-\infty}^{\infty} |h\alpha_h| < \infty$, the limiting distribution of $n^{1/2}(S^2 - \sigma^2)$ is normal with mean zero and variance $(\alpha_{4,\varepsilon} - 3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2$ (Fuller, 1996, p. 333). That is, as m and n/m go to infinity simultaneously,

$$\sqrt{n}(S^2 - \sigma^2) \xrightarrow{D} N\Big(0,\ (\alpha_{4,\varepsilon}-3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2\Big)$$

and

$$\sqrt{m}(S_j^2 - \sigma^2) \xrightarrow{D} N\Big(0, (\alpha_{4,\varepsilon}-3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2\Big).$$

Here "$\xrightarrow{D}$" means convergence in distribution. Since batch variances are asymptotically independent and normally distributed, as n and n/m are large,

$$\frac{m\sum_{j=1}^{n/m}(S_j^2 - S^2)^2}{(\alpha_{4,\varepsilon}-3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2} \sim \chi^2_{\frac{n}{m}-1},$$

where χ^2_ν denotes the chi-square distrubituion with ν degrees of freedom. Therefore,

$$\mathrm{Var}[\widehat{V}^{(NBV)}] = \mathrm{Var}\left[\frac{m^2}{n(n-m)}\sum_{j=1}^{n/m}(S_j^2 - S^2)^2\right]$$

$$\approx \frac{m^4}{n^2(n-m)^2} \cdot \frac{\left[(\alpha_{4,\varepsilon}-3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2\right]^2}{m^2} \cdot \mathrm{Var}\left[\frac{m\sum_{j=1}^{n/m}(S_j^2 - S^2)^2}{(\alpha_{4,\varepsilon}-3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2}\right]$$
$$= \frac{2m}{n^2(n-m)}\left[(\alpha_{4,\varepsilon}-3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2\right]^2.$$

Recall that $R_0^2 = \left(\sigma_\varepsilon^2 \sum_{j=-\infty}^{\infty} \alpha_j^2\right)^2$ in Equation (12), and from Anderson (1994, p. 468), we have $\left(\sigma_\varepsilon^2 \sum_{j=-\infty}^{\infty} \alpha_j^2\right)^2 = \sigma_\varepsilon^4 \sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \alpha_j^2 \alpha_{j+h}^2$. Therefore, as $m \longrightarrow \infty$ and $n/m \longrightarrow \infty$,

$$\begin{aligned}
\lim_{\substack{m\to\infty \\ n/m\to\infty}} \frac{n^3}{m}\mathrm{Var}[\widehat{V}^{(NBV)}] &= 2\left[(\alpha_{4,\varepsilon}-3)R_0^2 + 2\sum_{h=-\infty}^{\infty} R_h^2\right]^2 \\
&= 2\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4 \sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \alpha_j^2\alpha_{j+h}^2 + 2\sum_{h=-\infty}^{\infty} R_h^2\right]^2 \\
&= 2\left\{\sum_{h=-\infty}^{\infty}\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4 \sum_{j=-\infty}^{\infty} \alpha_j^2\alpha_{j+h}^2 + 2R_h^2\right]\right\}^2 \\
&= 2\left\{\sum_{h=-\infty}^{\infty}\left[(\alpha_{4,\varepsilon}-3)\sigma_\varepsilon^4 \sum_{j=-\infty}^{\infty} \alpha_j^2\alpha_{j+h}^2 + 2R_h^2 + R_0^2 - R_0^2\right]\right\}^2 \\
&= 2\left\{\sum_{h=1}^{\infty}\left(\mathrm{E}[X_i^2X_{i+h}^2] - \mathrm{E}[X_i^2]\mathrm{E}[X_{i+h}^2]\right)\right\}^2 \\
&= 2\left[\sum_{h=-\infty}^{\infty}\mathrm{Cov}(X_i^2, X_{i+h}^2)\right]^2 = 2\left[\sum_{h=-\infty}^{\infty}\widetilde{\rho}_h\ \mathrm{Var}(X_i^2)\right]^2 \\
&= 2\left[\widetilde{\gamma}_0(\alpha_4-1)\sigma^4\right]^2. \qquad \blacksquare
\end{aligned}$$

Results 2 and 3 show that for linear processes, the bias and variance of the NBV estimator go to zero at rates nm and n^3/m, respectively. These convergence rates are the same as the NBM estimator rates. Schmeiser et al. (1990) also provide sufficient conditions to ensure the unbiasedness of batch statistics. Often, these conditions hold only in the limit as the batch size m and number b of batches grow large.

3.1 Results of NBV Applied to OBV

Results 2 and 3 show that NBV behaves in a manner similar to NBM. It follows that NBV applied to the data process $\{X_i\}$ is equivalent to NBM applied to the data process $\{(X_i - \bar{X})^2\}$. Since OBM is essentially the same as NBM except for the overlapping algorithm, it applies to the data process $\{(X_i - \bar{X})^2\}$ as well as the data process $\{X_i\}$. Meketon and Schmeiser (1984) and Song and Schmeiser (1995) established that for the same assumptions and batch size, asymptotically OBM has the same mean as NBM and only 2/3 the asymptotic variance because of different batching logic. Similarly, NBM and OBM satisfy the asymptotic relationships:

$$\lim_{\substack{m\to\infty \\ m/n\to 0}} \frac{\mathrm{Bias}[\widehat{V}^{(OBV)}]}{\mathrm{Bias}[\widehat{V}^{(NBV)}]} = 1 \qquad \text{and} \qquad \frac{\mathrm{Var}[\widehat{V}^{(OBV)}]}{\mathrm{Var}[\widehat{V}^{(NBV)}]} = \frac{2}{3}\,. \tag{20}$$

Moreover, based on the limiting bias and variance of NBV in Equations (16) and (17), it follows that

$$\lim_{\substack{m\to\infty \\ m/n\to 0}} nm\ \mathrm{Bias}[\widehat{V}^{(OBV)}] \;=\; -\widetilde{\gamma}_1(\alpha_4 - 1)\sigma^4, \tag{21}$$

and

$$\lim_{\substack{m\to\infty \\ m/n\to 0}} \frac{n^3}{m}\ \mathrm{Var}[\widehat{V}^{(OBV)}] \;=\; \frac{4}{3}[\widetilde{\gamma}_0(\alpha_4 - 1)\sigma^4]^2. \tag{22}$$

An empirical study in Ceylan (1994) conforms closely to Equations (20) to (22). Yeh[a] (2002) also proves that the OBV results are valid for the MA(1) process.

3.2 The Asymptotically Optimal Batch Size

Using the asymptotic bias and variance results, we can compute the optimal batch size that minimizes the asymptotic mse. For large values of m and n/m, the bias and variance of NBV and OBV estimators are approximately

$$\mathrm{Bias}[\widehat{V}^{(NBV)}] \;\approx\; -(nm)^{-1}\widetilde{\gamma}_1(\alpha_4 - 1)\sigma^4,$$

$$\mathrm{Var}[\widehat{V}^{(NBV)}] \approx 2mn^{-3}[\widetilde{\gamma}_0(\alpha_4 - 1)\sigma^4]^2,$$

and

$$\begin{aligned}\mathrm{Bias}[\widehat{V}^{(OBV)}] &\approx -(nm)^{-1}\widetilde{\gamma}_1(\alpha_4 - 1)\sigma^4, \\ \mathrm{Var}[\widehat{V}^{(OBV)}] &\approx (4/3)mn^{-3}[\widetilde{\gamma}_0(\alpha_4 - 1)\sigma^4]^2.\end{aligned}$$

Therefore, the mse for NBV and OBV, respectively, is approximately

$$\mathrm{mse}[\widehat{V}^{(NBV)}] \approx \left(\frac{\tilde{\gamma}_1^2}{n^2m^2} + \frac{2m\tilde{\gamma}_0^2}{n^3}\right)\left[(\alpha_4 - 1)\sigma^4\right]^2, \tag{23}$$

$$\mathrm{mse}[\widehat{V}^{(OBV)}] \approx \left(\frac{\tilde{\gamma}_1^2}{n^2m^2} + \frac{4m\tilde{\gamma}_0^2}{3n^3}\right)\left[(\alpha_4 - 1)\sigma^4\right]^2. \tag{24}$$

The asymptotic optimal batch sizes $\widehat{m}^*$ that minimize Equations (23) and (24) are listed in Table 1.

Table 1 summarizes the asymptotic results for batch-mean and batch-variance estimators. Because the batch-mean estimator is the batch-variance estimator applied to the data $\{(X_i - \bar{X})^2\}$, which converges to the process $\{(X_i - \mu)^2\}$, substitution of $\mathrm{Var}[(X_i - \mu)^2] = (\alpha_4 - 1)\sigma^4$ for σ^2, $\widetilde{\gamma}_0$ for γ_0, and $\widetilde{\gamma}_1$ for γ_1 in the batch-mean asymptotic results yields the batch-variance asymptotic results. Notice that the asymptotically optimal batch sizes for batch-mean and batch-variance estimators are different because the values of γ_i and $\tilde{\gamma}_i$ are different for $i = 0$, 1.

HERE IS TABLE 1

4 Conclusions

This paper concerns the measurement of performance via the sample variance, an estimate of the population variance, with the denominator being the number of observations and the data being correlated and identically distributed. In contrast to the estimation of the variance of the sample mean, which has been the topic of intensive research, the variance of the sample variance has received little attention. The batching method estimates the variance of the sample variance by dividing the observations into several batches, computing the batch variances (the sample variances for each batch), and computing the estimator as a function of the batch variances. Two variants of the estimator are considered: the NBV (for non-overlapping batches) and the OBV (for overlapping batches) .

We provide analytical results for NBV estimators, assuming that the data are from a linear process. By viewing the sample variance as a sample mean of the squared terms, we show that the asymptotic results for the batch-variance method and the batch-mean method are analogous. We consider three aspects of the asymptotic results: convergence rates; constant multipliers; and data properties, which determine the second multiplier. We show that 1. Both methods have the same convergence rates and constant multipliers, and 2. Because the data properties are analogous, one principle applies consistently to the second multiplier in the batch-mean and batch-variance results. Moreover, both of these asymptotic results can be extended to OBV estimators. If the OBV to NBV asymptotic bias and variance ratios for batch variances are the same as for batch means, the OBV and OBM estimators behave analogously. Specifically, the OBV bias is the same as the NBV bias and the OBV variance is only 2/3 of the NBV variance.

This paper points to future work in three areas: (i) estimation of the asymptotically optimal batch size (a function of unknown $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$) for NBV and OBV estimators, (ii) extension of our results to batch statistics that can be viewed as a batch-mean estimator (e.g., quantile estimators), and (iii) development of proofs for the OBV results.

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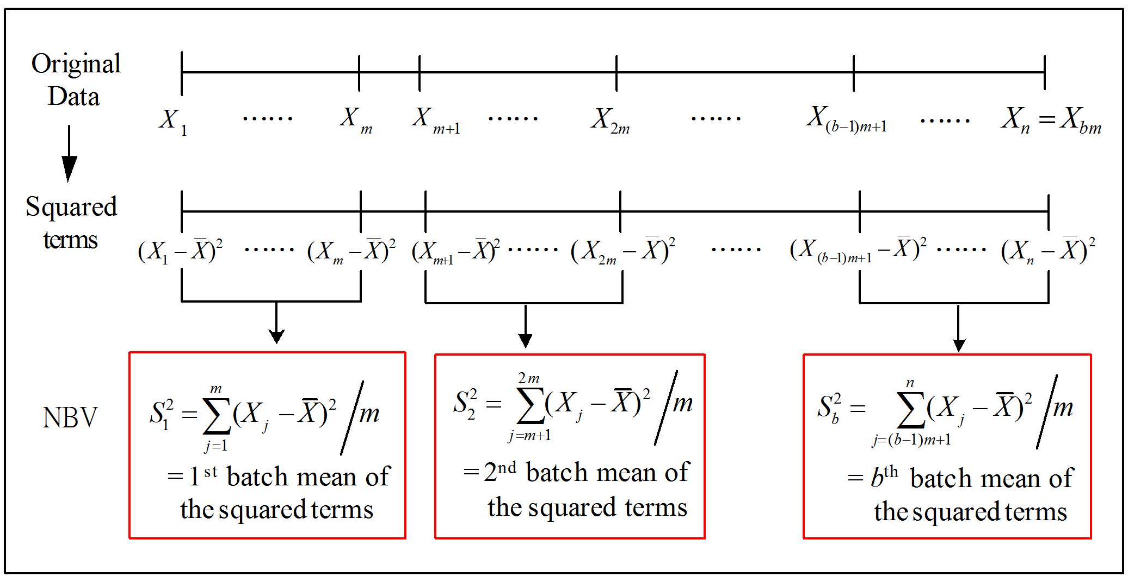
Table 1. Comparisons of batch-mean and batch-variance estiamtors

Figure 1. Diagram of the NBV method

	Estimator Type			
Property	NBM	OBM	NBV	OBV
nmBias	$-\sigma^2\gamma_1$	$-\sigma^2\gamma_1$	$-(\alpha_4-1)\sigma^4\widetilde{\gamma}_1$	$-(\alpha_4-1)\sigma^4\widetilde{\gamma}_1$
n^3m^{-1}Var	$2\left(\sigma^2\gamma_0\right)^2$	$\frac{4}{3}\left(\sigma^2\gamma_0\right)^2$	$2[(\alpha_4-1)\sigma^4\widetilde{\gamma}_0]^2$	$\frac{4}{3}[(\alpha_4-1)\sigma^4\widetilde{\gamma}_0]^2$
$\hat{m}^*$	$1+\left[n\left(\frac{\gamma_1}{\gamma_0}\right)^2\right]^{1/3}$	$1+\left[\frac{3n}{2}\left(\frac{\gamma_1}{\gamma_0}\right)^2\right]^{1/3}$	$1+\left[n\left(\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}_0}\right)^2\right]^{1/3}$	$1+\left[\frac{3n}{2}\left(\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}_0}\right)^2\right]^{1/3}$


Original Data
X_1 ...... X_m X_{m+1} ...... X_{2m} ...... $X_{(b-1)m+1}$ ...... $X_n = X_{bm}$
Squared terms
$(X_1-\bar{X})^2$ ...... $(X_m-\bar{X})^2$ $(X_{m+1}-\bar{X})^2$ ...... $(X_{2m}-\bar{X})^2$ ...... $(X_{(b-1)m+1}-\bar{X})^2$ ...... $(X_n-\bar{X})^2$
NBV
$S_1^2 = \sum_{j=1}^{m}(X_j-\bar{X})^2 \Big/ m$
$= 1^{st}$ batch mean of the squared terms
$S_2^2 = \sum_{j=m+1}^{2m}(X_j-\bar{X})^2 \Big/ m$
$= 2^{nd}$ batch mean of the squared terms
$S_b^2 = \sum_{j=(b-1)m+1}^{n}(X_j-\bar{X})^2 \Big/ m$
$= b^{th}$ batch mean of the squared terms