Computation of the Sample Size and Coverage for Guaranteed-Coverage Nonnormal Tolerance Intervals

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ABSTRACT

We propose Monte Carlo algorithms to estimate the sample size and coverage of guaranteed-coverage tolerance intervals for nonnormal distributions. The current literature focuses on computation of the tolerance factor, but addresses less on the sample size, coverage, and confidence, which need to be set prior to the tolerance factor. The coverage estimation algorithm, which always converges, is based on our proof that the coverage is a quantile of an observable random variable. The sample-size estimation algorithm, which seems to converge in empirical results if the root is unique, is based on the general stochastic root-finding algorithm, retrospective approximation. Following previous sensitivity analysis for the tolerance factor, we analyze relationships among the sample size, coverage, and confidence.

KEYWORDS: Quantile, Reliability, Retrospective Approximation, Stochastic Root Finding.

1 INTRODUCTION

We consider guaranteed-coverage tolerance intervals (GCTIs) for random product characteristic X whose distribution F_X is continuous but has unknown mean μ and unknown variance σ^2. Based on a random sample $\{X_1, X_2, ..., X_n\}$ from the distribution F_X, a GCTI for X is defined as $I(\bar{X}, S, k)$, where $I(\bar{X}, S, k)$ equals $(\bar{X} - k\,S, \infty)$ for lower one-sided, $(-\infty, \bar{X} + k\,S)$ for upper one-sided, and $(\bar{X} - k\,S, \bar{X} + k\,S)$ for two-sided intervals (Wald and Wolfowitz, 1946), where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. For such intervals, a practitioner can state with confidence γ that the proportion of the population in the tolerance interval $I(\bar{X}, S, k)$, based on a sample of size n, is at least α. The four tolerance parameters—sample size $n \in \{2, 3, ...\}$, tolerance factor $k \in R$, coverage $\alpha \in (0, 1)$, and confidence $\gamma \in (0, 1)$—are determined so that

$$\Pr_{\bar{X},S}\{ \Pr_X\{ X \in I(\bar{X}, S, k) \} \geq \alpha \} = \gamma. \tag{1}$$

GCTIs have wide uses in quality control and system reliability. For more examples, see Chen and Schmeiser (1995), Patel (1986), and Odeh and Owen (1980).

The existing literature focuses on computation of the tolerance factor k. Most of it assumes normal distributions, e.g., Wald and Wolfowitz (1946), Guttman (1970), Aitchison and Dunsmore (1975), Odeh and Owen (1980), and Eberhardt et al. (1989). The one-sided tolerance factor for normal distributions is a multiple of a noncentral t quantile (see Section 2). The two-sided factor can be computed by solving a nonlinear equation. Chen and Schmeiser (1995) propose quantile estimation methods to compute the tolerance factor for nonnormal parametric distributions (Section 2). Other tolerance-interval studies for nonnormal distributions include Aitchison and Dunsmore (1975) and Patel (1986), who discuss different forms of tolerance intervals for binomial, Poisson, exponential, gamma, and other standard populations. Guenther (1985) provides an extensive discussion of distribution-free tolerance intervals.

Before computation of the tolerance factor, values of n, α, and γ need to be set. Consider a reliability program adopted by a rocket manufacturer. The tested item is considered defective if its characteristic measurement falls below the tolerance limit $\bar{X} - kS$, based on a random sample of size n. The reliability engineer wants to state with γ confidence that the system reliability is at least α; that is, with confidence γ, the probability that the characteristic measurement falls below the tolerance limit is less than $(1 - \alpha)$. To achieve the desired values of the minimum reliability α and confidence γ, the sample size n must be chosen appropriately. The following "sample-size determination procedure" is used to determine the sample size. To search for the sample size, the tentative tolerance factor k is set so that the tolerance limit $\bar{X} - kS$ equals the lower specification limit L (the minimum value required).

Sample-size determination procedure: Given coverage α and confidence γ:

1. Compute the sample size n satisfying Equation 1 for a given nominal value of k.
2. Collect a sample $\{x_1, ..., x_n\}$ (from the real system, instead of the Monte Carlo experiment) and compute the tolerance factor

$$k = (\bar{x} - L)/s,$$

   where $\bar{x} = \sum_{i=1}^{n} x_i/n$, $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)$, and constant L is the lower specification limit of product characteristic.

3. Compute the coverage $\tilde{\alpha}$ so that with confidence γ the interval $(\bar{X} - kS, \infty)$ contains at least proportion $\tilde{\alpha}$ of the population, i.e., solve the following equation for $\tilde{\alpha}$:

$$\Pr_{\bar{X},S}\{\ \Pr_X\{\ X \geq \bar{X} - kS)\ \} \geq \tilde{\alpha}\ \} = \gamma. \tag{2}$$

4. If $\tilde{\alpha} > \alpha$, stop and return n; otherwise, update n and go to Step 2.

Our motivation for computing the sample size and coverage for nonnormal distributions arises from the one-dimensional root-finding problems in Steps 1 and 3. Step 1 computes the sample size n satisfying Equation 1, given k, α, γ, and the distribution shape (e.g., beta); Step 3 computes the coverage $\tilde{\alpha}$ satisfying Equation 2 (or α for Equation 1), given γ, n, k, and the distribution shape. A traditional approach to these two problems is to build an extensive table of values for the four tolerance parameters satisfying Equation 1 for each distribution of interest. With such a table, the practitioner can choose values of the sample size and coverage from the table whenever they are needed. The drawbacks to this approach are that any table with a wide range of tolerance-parameter values and distribution types would be huge, and that interpolation—or worse, extrapolation—may be used to approximate desired values that are not listed in the table.

In this research, we are interested in black-box Monte Carlo algorithms that compute any tolerance parameter of interest. That is, we consider the problem of finding any lower one-sided tolerance parameter (e.g., n) that satisfies the tolerance logic (Equation 1) when the other three parameters (e.g., k, α, and γ) and the distribution shape are known. Specifically, the research problem is as follows:

Research Problem: Given

(a) the shape of continuous distribution function $F_X()$ with unknown mean μ and unknown variance σ^2,

(b) three of the following four tolerance parameters:

- sample size n,
- tolerance factor k,
- coverage α,
- confidence γ.

Find: the other unknown tolerance parameter satisfying the tolerance logic, i.e.,

$$\Pr_{\bar{X},S}\{ \Pr_X\{ X \geq \bar{X} - k\,S \} \geq \alpha \} = \gamma. \tag{3}$$

The assumption that the distribution shape is known means that all standardized moments are known but the mean or variance is unknown. For example, the reliability engineer may model the product-characteristic distribution as a beta distribution with unknown mean and unknown variance. In case the real data do not adequately fit any standard distributions (e.g. distributions usually presented in introductory Statistics textbooks), practitioners can define the distribution through a simulation routine that can generate observations if the mean and variance are also specified. (See Law and Kelton, 1991, page 462, for random-variate generation methods.) This is valid because our solution approaches are based on the Monte Carlo sampling. In this paper, we focus on estimation algorithms for lower one-sided GCTIs. The proposed algorithms in Section 3 can be modified easily for upper one-sided GCTIs and extended for two-sided GCTIs. Furthermore, we propose algorithms only for the sample size and the coverage. The tolerance factor k can be estimated by the quantile estimation methods proposed in Chen and Schmeiser (1995). Furthermore, estimating the confidence γ is straightforward (Schmeiser, 1990). We discuss the methods of computing k and γ in Section 2.

The rest of this paper is organized as follows: In Section 2, we review the related literature. In Section 3, we propose Monte Carlo estimation algorithms for the coverage and sample size. The coverage estimator always converges; the sample-size estimator seems to converge in our simulation results if the root is unique. In Section 4, we continue Chen and Schmeiser's (1995) sensitivity analysis for the sample size, coverage, and confidence. In Section 5, we illustrate an example to show the application of our coverage and sample-size computation software.

2 LITERATURE REVIEW

This section reviews the literature on computation of the four tolerance parameters. The confidence estimator and tolerance-factor estimator discussed here are designed for nonnormal parametric distributions and the coverage estimator is for normal distributions. The literature on sample size is different from, but related to, our problem. We discuss the computation of these four tolerance parameters in turn.

(1) Confidence:

We consider the problem of computing the confidence that the tolerance interval $[\bar{X} - kS, \infty)$ contains at least proportion α of the measurements for nonnormal distributions. That is, computing the probability $\gamma = \Pr_{\bar{X},S}\{ \Pr_X\{ X \geq \bar{X} - k\, S \} \geq \alpha \}$, given n, k, α, and the shape of distribution F_X. Numerical computation of γ may not be efficient because γ is a $(n+1)$-dimensional integral (except for special cases like normal distributions, in which γ is a noncentral t percentage point). However, γ can be estimated easily by Monte Carlo simulation. We can generate m samples $\{x_{11}, ..., x_{1n}\}$,..., $\{x_{m1}, ..., x_{mn}\}$ from the distribution F_X, using any arbitrary values of μ and σ. (Notice that γ does not depend on the unknown mean μ or standard deviation σ of distribution F_X.) For each sample, we compute the sample mean $\bar{x}_i$ and sample standard deviation s_i of $\{x_{i1}, ..., x_{in}\}$, $i = 1, ..., m$. Then the confidence γ can be estimated by

$$\hat{\gamma} = \sum_{i=1}^{m} y_i/m, \tag{4}$$

where

$$y_i = \begin{cases} 1 & \text{if } \Pr_X\{X \geq \bar{x}_i - k\, s_i\} \geq \alpha \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to show that the estimate $\hat{\gamma}$ is unbiased with variance $\gamma(1-\gamma)/m$.

(2) Tolerance Factor:

Chen and Schmeiser (1995) propose quantile estimation methods for computing the tolerance factor for nonnormal distributions, as defined earlier in the research problem. They show that the tolerance factor k satisfying Equation 3 is the γth quantile of the random variable $K = [\bar{X} - F_X^{-1}(1-\alpha)]/S$. This result follows from the equivalence of Equation 3 and $\Pr_K\{K \leq k\} = \gamma$. Notice that K is observable because K does not depend on the population mean μ or standard deviation σ. Therefore we can generate m samples of size n from distribution F_X, using any arbitrary values of μ and σ, and compute the sample mean $\bar{x}_i$ and sample standard deviation s_i of the ith sample, $i = 1, ..., m$. Let $k_i = [\bar{x}_i - F_X^{-1}(1-\alpha)]/s_i$; then we have m realizations k_1,...,k_m of the random variable K. Hence the tolerance factor estimate is $\hat{k} = \omega\ k_{(\lfloor (m+1)\gamma \rfloor)} + (1-\omega)\ k_{(\lceil (m+1)\gamma \rceil)}$, the convex combination of the $\lfloor (m+1)\gamma \rfloor$th and $\lceil (m+1)\gamma \rceil$th order statistics, where the weight is $\omega = \lceil (m+1)\gamma \rceil - (m+1)\gamma$. (Here, $\lfloor a \rfloor$ is the biggest integer less than or equal to a and $\lceil a \rceil$ is the smallest integer greater than or equal to a.) The asymptotic distribution $\sqrt{m}(\hat{k} - k)$ is normal with mean 0 and variance $\gamma(1-\gamma)/[f_K^2(k)]$, where $f_K(\cdot)$ is the density function of the random variable K. (See Lehmann, 1983, p 394.)

If the distribution F_X is normal, then random variable $\sqrt{n}K$ is a noncentral t with $n-1$ degrees of freedom and noncentrality parameter $\sqrt{n}z_\alpha$, where z_α is the αth quantile of the standard normal. Therefore, the tolerance factor is given by $k = t_{n-1,\gamma}(\sqrt{n}z_\alpha)\ /\ \sqrt{n}$, where $t_{\nu,\gamma}(\delta)$ is the γth quantile of the noncentral t with ν degrees of freedom and noncentrality δ. For this special case, numerical computation of k would be more efficient.

(3) Coverage:

Owen and Hua (1977) derive numerical methods for computing the coverage for normal distributions, i.e., solving Equation 3 for α, given n, k, γ, and the normal distribution shape. The purpose is to obtain the γ-confidence limit (i.e., α) for the random coverage $\Pr_X\{X \geq \bar{X} - kS\}$. They use the result $\sqrt{n}k = t_{n-1,\gamma}(\sqrt{n}z_\alpha)$ and suggest searching for the noncentrality $\sqrt{n}\alpha$ so that the noncentral t (i.e., $\sqrt{n}K$) percentage point at $\sqrt{n}k$ is γ. However, they did not mention any search methods. Odeh and Owen (1980) provide tables of α values, which are computed by Newton's method (see page 274).

In the case of nonnormal distributions, the random variable $\sqrt{n}K$ may not have noncentral t

or other standard distributions. Numerical evaluation of α is difficult. In Section 3.1, we propose a quantile estimation approach that requires no root search.

(4) Sample Size:

Most literature on the sample size assumes normal distributions and addresses a problem slightly different from ours. In additional to the criterion of Equation 3, another criterion is added so that the tolerance interval does not cover too large a proportion of the population, relative to the lower limit α. When the sample size is very small, the interval becomes very wide and is of little use. Faulkenberry and Weeks (1968), Faulkenberry and Daly (1970), and Kirkpatrick (1977) suggest a second criterion of $\Pr_{\bar{X},S}\{\ \Pr_X\{\ X \geq \bar{X} - k\,S\ \} \geq \alpha'\ \} = \delta$, where $\alpha' > \alpha$ and δ is small. Then, the confidence that the random coverage $\Pr_X\{\ X \geq \bar{X} - k\,S\ \}$ lies between α and α' is $(\gamma - \delta)$. Other variations of the second criterion include controlling both limits of the random coverage around α (Wilks, 1941, and Odeh et al., 1989) and controlling coverage on both tails of two-sided intervals (Chou and Mee 1984). Since there are two criteria (i.e., two equations), this procedure determines not only the sample size but also the tolerance factor, while α and γ are pre-chosen values.

3 METHODS

In Sections 3.1 and 3.2, we propose algorithms to solve Equation 3 for the nonnormal coverage α and for the nonnormal sample size n, respectively. To estimate α, we invert the root-finding equation to show that α is a quantile of an observable random variable and then estimate α using order statistics. To estimate n, we apply retrospective approximation algorithms developed by Chen and Schmeiser (1994b), with small changes for the discrete root-finding function of the sample size n. We show that the estimate of α always converges and that if the root is unique, the estimate of n seems to converge in our simulation results.

3.1 Computing the coverage

We consider finding the coverage α for nonnormal distributions, such that the tolerance interval contains at least proportion α of the population, with confidence γ. That is, we solve Equation 3 for α, given values of n, k, γ, and the distribution shape. We propose an estimation method similar to the quantile estimation method for the tolerance factor k (Section 2). Analogous to the random variable K for the tolerance factor, we define the random variable $C = \Pr_X\{ X \geq \bar{X} - k\,S \}$, the random coverage of the tolerance interval. Then Equation 3 is equivalent to the equation

$$\Pr_C\{C \geq \alpha \} = \gamma.$$

Hence α is the $(1-\gamma)$th quantile of the random variable C. Again, the random variable C is observable because it does not depend on the mean μ or standard deviation σ of distribution F_X. Therefore, we can generate m independent Monte Carlo observations of C, using arbitrary values of μ and σ. Then we can estimate α by

$$\hat{\alpha} = \omega\, C_{(\lfloor (m+1)(1-\gamma)\rfloor)} + (1-\omega)\, C_{(\lceil (m+1)(1-\gamma)\rceil)}, \tag{5}$$

the convex combination of the $\lfloor (m+1)(1-\gamma)\rfloor$th and $\lceil (m+1)(1-\gamma)\rceil$th order statistics, where the weight is $\omega = \lceil (m+1)(1-\gamma)\rceil - (m+1)(1-\gamma)$. Specifically, the algorithm performs as follows.

Given: sample size n, tolerance factor k, confidence γ, and the distribution shape.

Procedure:

1. Independently generate m random samples $\{x_{11},...,x_{1n}\}$,..., $\{x_{m1},...,x_{mn}\}$ from the distribution F_X using any arbitrary values of μ and σ.
2. Compute the sample mean $\bar{x}_i$ and standard deviation s_i for $i = 1,...,m$.
3. Compute $c_i = \Pr_X\{ X \geq \bar{x}_i - k\,s_i \}$ for $i = 1,...,m$.
4. Compute $\hat{\alpha}$ from $c_1, c_2, ..., c_m$ using Equation 5.

By Lehmann (1983, p. 394), the asymptotic distribution of $\hat{\alpha}$ is

$$\sqrt{m}(\hat{\alpha} - \alpha) \rightarrow N(0, \frac{\gamma(1-\gamma)}{f_C^2(\alpha)}), \tag{6}$$

where $f_C(\cdot)$ is the density function of C. Hence, the estimator $\hat{\alpha}$ always converges at rate $O(1/\sqrt{m})$.

In the case of a normal population, the probability $\Pr_C\{C \leq \alpha\}$ is the cumulative probability at point $\sqrt{n}k$ of a noncentral t with $n-1$ degrees of freedom and noncentrality parameter $\sqrt{n}z_\alpha$. For this special case, α can be computed numerically such that with the noncentrality $\sqrt{n}z_\alpha$, the noncentral-t cumulative probability at $\sqrt{n}k$ is γ (See Section 2, (3)).

3.2 Computing the Sample Size

Here we consider finding the sample size n for nonnormal distributions such that a practitioner can state with confidence γ that the tolerance interval $[\bar{X} - kS, \infty)$, based on a sample of size n, contains at least the proportion α of the population. That is, we solve Equation 3 for the root n, given values of k, α, γ, and the distribution shape. Unlike in Section 3.1, it is difficult here to invert the root-finding function to express n as a statistical constant, e.g., quantile. We implement the general stochastic root-finding algorithm, retrospective approximation (RA), with small modifications. Our simulation results show that when the root is unique, the modified RA seems to converge to the root despite lack of convergence proof. To emphasize their dependence on the sample size n, we denote the sample mean $\bar{X}$ and sample standard deviation S by $\bar{X}_n$ and S_n. Furthermore, for convenience, we denote the root-finding function of any sample size $\tilde{n}$ by $g(\tilde{n}; k, \alpha) = \Pr_{\bar{X}_{\tilde{n}}, S_{\tilde{n}}}\{\ \Pr_X\{\ X \geq \bar{X}_{\tilde{n}} - k\, S_{\tilde{n}}\ \} \geq \alpha\ \}$, the confidence that the random coverage is at least α. It follows that Equation 3 is equivalent to the equation $g(n; k, \alpha) = \gamma$. We want to solve this equation for n. For the case of normal population, the sample size n can be computed numerically. As mentioned in Section 3.1, $g(n; k, \alpha)$ is the percentage point $\Pr\{T_{n-1}(\sqrt{n} z_\alpha) \leq \sqrt{n} k\}$ of the noncentral t random variable $T_{n-1}(\sqrt{n} z_\alpha)$ with $n - 1$ degrees of freedom and noncentrality parameter $\sqrt{n} z_\alpha$. For this special case, the sample size n can be computed numerically by searching for the degrees of freedom $(n - 1)$ such that the noncentral-t percentage point at $\sqrt{n} k$ is γ.

Knowing properties of the root-finding function is useful for solving the equation. The function g has four properties: (1) Function $g(\tilde{n}; \cdot, \cdot)$ is discrete because the sample size $\tilde{n} \in \{2, 3, 4, ...\}$. (2) Function $g(\tilde{n}; \cdot, \cdot)$ is nonmonotonic with respect to $\tilde{n}$, even if F_X is normal. Since confidence γ increases with the tolerance factor k but k is not necessarily monotonic with the sample size n (Chen and Schmeiser, 1995), the confidence is not monotonic with the sample size. Hence, $g(\tilde{n}; \cdot, \cdot)$ is not monotonic. Figures 1(a) and 1(b) illustrate two nonmonotonic functions g, respectively: $g(\tilde{n}; -1.2067, .1)$ for the normal distribution and $g(\tilde{n}; .4125, .99)$ for the Johnson S_B distribution with skewness 4 and kurtosis 30. The Johnson distribution family, proposed by Johnson (1949), has three transformations of the standard normal distribution, resulting in lognormal, bounded (denoted as S_B), and unbounded (denoted as S_U) distributions (see Appendix). From Figure 1(a) we see that the function $g(\tilde{n}; -1.2067, .1)$ with normal distribution decreases with $\tilde{n}$ first and then increases. Figrue 1(b) shows that $g(\tilde{n}; .4125, .99)$ is convex for smaller $\tilde{n}$ and concave for larger $\tilde{n}$.

Figure 1: Plot of Two Nonmonotonic Functions $g(\tilde{n}; k, \alpha)$: In (a), $k = -1.2067$, $\alpha = 0.1$, and the distribution shape is normal; in (b), $k = .4125$, $\alpha = 0.99$, and the distribution shape is Johnson S_B with skewness 4 and kurtosis 30

(3) The limiting value of $g(\tilde{n}; k, \alpha)$ is

$$\lim_{\tilde{n}\to\infty} g(\tilde{n}; k, \alpha) = \begin{cases} 1 & \text{if } k \geq k^{\infty} \\ 0 & \text{otherwise} \end{cases},$$

where $k^{\infty} = [\mu - F_X^{-1}(1-\alpha)]/\sigma$. (See Figure 3.) (4) The root n may not be unique or may not exist, even for the normal population. For example, if $F_X(\cdot)$ is symmetric at mean, $g(\tilde{n}; 0, 0.5)$ equals 0.5 for any sample size $\tilde{n}$. Therefore, the equation $g(\tilde{n}; 0, 0.5) = \gamma$ has an infinite number of roots if $\gamma = 0.5$, and has no root, otherwise. Figure 1 (b) also shows another example of multiple roots: the equation $g(n; .4125, .99) = .001$ has two roots 10 and 71.

Solving equation $g(\tilde{n}; k, \alpha) = \gamma$ for the root $\tilde{n} = n$ is a stochastic root-finding problem (SRFP, Chen and Schmeiser, 1994a), solving a deterministic equation using only estimates of function values. As mentioned in Section 2, the function value $g(\tilde{n}; k, \alpha)$ is an $(\tilde{n}+1)$ dimensional integral, and therefore numerical computation of g may not be efficient (except for special cases like normal distributions). However, we can estimate the function value easily via simulation experiment. Equation 4 shows an unbiased estimator $\hat{g}(\tilde{n}; k, \alpha) = \sum_{i=1}^{m} Y_i/m$ of $g(\tilde{n}; k, \alpha)$, where Y_i equals 1 if $\Pr_X\{X \geq \bar{X}_i - kS_i\} \geq \alpha$ and equals 0, otherwise, for $i = 1, ..., m$.

Chen and Schmeiser (1994b) propose retrospective approximation (RA) algorithms for SRFPs, with continuous root-finding functions over the real line. Let $\hat{g}(\tilde{n}; k, \alpha, \underline{\omega})$ denote the estimate

$\hat{g}(\tilde{n}; k, \alpha)$ generated from the simulation experiment using a vector of m pseudo-random number streams $\underline{\omega} = (\omega_1, ..., \omega_m)$. Each stream ω_i is used to generate the ith sample $\{x_{i1}, ..., x_{i\tilde{n}}\}$ from distribution F_X, where $i = 1, ..., m$. RA iteratively solves a sequence of sample-path equations $\{\hat{g}(N_i^*; k, \alpha, \underline{\omega}_i) = \gamma : i = 1, 2, ...\}$, where $\underline{\omega}_i = (\omega_{i1}, ..., \omega_{im_i})$ and the sequence $\{m_1, m_2, ...\}$ is increasing. In each iteration, the sample-path equation is solved until a bounding interval of the retrospective root N_i^* is found, starting at an initial point and moving by a step-size δ_i, which is doubled each time. The linear interpolate of the bounds, called N_i, is returned. After i iterations, the root estimator $\bar{N}_i$ is then the weighted average of those solutions $N_1, N_2, ..., N_i$, where the jth weight is proportional to the number of samples m_j for $j = 1, ..., i$. RA assumes that the root-finding function is continuous over the whole real line, lies below γ over the domain below the true root, and lies above γ over the domain above the root. Additional conditions on g and $\hat{g}$ guarantee that the RA root estimator converges to the true root with probability one (Chen, 1994). A specific RA version, called bounding independent RA (BIRA), performs as follows.

BIRA Algorithm:

Given algorithm parameters: the standard error tolerance σ_0, initial solution N_0, initial number of samples m_1, initial step size δ_1, the number-of-samples multiplier c_1, and the step-size multiplier c_2.

Find: the root n satisfying $g(n; k, \alpha) = \gamma$.

Step 0. Initialize the BIRA iteration number $i = 1$.

Step 1. Independently generate $\underline{\omega}_i$.

Step 2. Solve Equation $\hat{g}(N_i^*; k, \alpha, \underline{\omega}_i) = \gamma$ until a bounding interval of the root N_i^* is found, starting at the point $\bar{N}_{i-1}$ (note: $\bar{N}_0 = N_0$) and moving by step size δ_i, which is doubled each time. Return the linear interpolate N_i of the bounds.

Step 3. Compute $\bar{N}_i = (\sum_{j=1}^{i} m_j)^{-1} \sum_{j=1}^{i} m_j N_j$ and its standard error estimate $\hat{se}(\bar{N}_i) = \sigma_N / \sqrt{\sum_{j=1}^{i} m_j}$, where σ_N equals $\sqrt{(i-1)^{-1}[\sum_{j=1}^{i} m_j N_j^2 - (\sum_{j=1}^{i} m_j)\bar{N}_i^2]}$ if $i \geq 2$, and 0 if $i = 1$.

Step 4. If $\hat{se}(\bar{N}_i) < \sigma_0$, stop. Otherwise, compute $\delta_{i+1} = \lfloor c_2 \sigma_N \sqrt{(\sum_{j=1}^{i} m_j)^{-1} + (m_{i+1})^{-1}} \rfloor$ (note $\delta_2 = \delta_1$) and $m_{i+1} = c_1 m_i$, let $i \longleftarrow i + 1$, and go to Step 1.

Algorithm BIRA is modified to solve the sample size problem. First, in the case where $k < k^\infty$, the root-finding function g is redefined as $1 - g$ and the target confidence γ is therefore changed to $1-\gamma$. Recall that BIRA assumes that g is under γ when the sample size is less than n and above γ otherwise. When $k < k^{\infty}$, the limiting value of $g(\tilde{n};\cdot,\cdot)$ approaches 0 as the sample size $\tilde{n}$ goes to infinity; therefore we suspect the function g is below γ when the sample size is greater than n. Redefining g as $1-g$ is to satisfy this BIRA assumption. However, the redefinition does not always work, e.g., for the function illustrated in Figure 1. Second, three rounding steps are added to each BIRA iteration i because the sample size and hence the function g are discrete: (1) Round down the step-size δ_i, since a small step size seems to be more efficient; (2) Round the retrospective solution N_i to the nearest integer; (3) Round up the root estimator $\bar{N}_i$ if $k \geq k^{\infty}$, and round down otherwise, to ensure that the confidence is at least γ.

The modified BIRA does not guarantee convergence though. Situations in which g is discrete, g might have multiple roots, and g might not satisfy the BIRA assumption of lying below γ for $\tilde{n} < n$ and above γ otherwise, make the root search complicated. Despite the lack of convergence proof, the modified BIRA algorithm seems to converge to the root in our simulation results when the root is unique. Even when the root is not unique, if the initial solution N_0 is well specified, BIRA might still converge to one of the roots.

To evaluate the performance of the modified BIRA, we run the simulation experiment with 66 design points, each for different combinations of distribution shape, α, γ and k. For the experiment, we choose the Johnson distribution family (see Appendix) for the distribution F_X. The skewness α_3 and kurtosis α_4, the third and fourth standardized moments, are used to measure the distribution shape, where each specified (α_3, α_4) corresponds to a unique Johnson distribution shape. We arbitrarily use $\mu = 0$ and $\sigma = 1$, since the sample size n is not a function of μ or σ. The 66 design points are:

- $(\alpha_3, \alpha_4) \in \{(0,3), (2,30), (4,30)\}$, corresponding to the normal, Johnson S_U, and Johnson S_B distribution shapes, respectively.
- α, γ, and k:

  For the normal distribution:

  1. $\alpha \in \{.1, .5, .9\}$ and $\gamma \in \{.1, .5, .9\}$, but excluding the combination $(\alpha, \gamma) = (.5, .5)$ because in this case k must be zero and therefore there are an infinite number of solutions (see the fourth property of g). We further delete half the combinations because k only changes sign when (α, γ) becomes $(1-\alpha, 1-\gamma)$; e.g., (.1, .9) and (.9, .1) have the same simulation performance. Therefore, only four (α, γ) combinations are included.

2. $k \in \{k_5, k_{50}, k_{500}\}$, where $k_{n'}$ is the tolerance factor corresponding to a root of n'; note that $k_{n'}$ is obtained by the quantile estimation method by Chen and Schmeiser (1995).

For the Johnson S_U and S_B distributions:

$$\alpha \in \{.001, .5, .99\}, \qquad \gamma \in \{.001, .5, .99\}, \qquad k \in \{k_2, k_{10}, k_{30}\}.$$

The simulation results for the sample size estimation are listed in Tables 1 and 2. Table 1 shows values of n and their estimates $\hat{n}$ for the normal distribution, where the estimates $\hat{n}$ are computed numerically. Table 2 shows simulation results for the Johnson S_B distribution with skewness 4 and kurtosis 30, and for the Johnson S_U distribution with skewness 2 and kurtosis 30. The estimates $\hat{n}$ in Table 2 are computed by the modified BIRA algorithm with 20 simulation runs. For each design point, Table 2 shows the true sample size n, the sample size estimate $\hat{n}$ $(= \bar{N}_i)$, its standard error estimate $\hat{se}(\hat{n})$, and the number of observations generated from the distribution F_X (denoted by *nobs*) to stand for the computation time. Only significant digits are listed. The BIRA algorithm parameters are set as follows: $m_1 = 10$, $\delta_1 = 1$, $c_1 = 2$, $c_2 = 1$, N_0 equals the normal n value for the given α, γ and k, and the standard error tolerance σ_0 is set so that the coefficient of variation (i.e., σ_0/n) equals 0.025. We use the coefficient of variation to eliminate the scaling problem since the estimate's variance tends to increase with its value.

Table 1: Sample-Size Estimators for the Normal Distribution

α	γ	k	n	$\hat{n}$	α	γ	k	n	$\hat{n}$
.1	.1	-2.7435	5	5	.1	.9	-.67525	5	5
.1	.1	-1.5594	50	50	.1	.9	-1.0594	50	50
.1	.1	-1.3618	500	500	.1	.9	-1.2067	500	500
.1	.5	-1.3818	5	5	.5	.1	-.68567	5	5
.1	.5	-1.2891	50	50	.5	.1	-.18372	50	50
.1	.5	-1.2823	500	499	.5	.1	-.05738	500	500

The two tables illustrate that the estimates $\hat{n}$ are very close to the true root n for most design points. For the normal distribution in Table 1, the numerical error is negligible. For the Johnson S_B and Johnson S_U distributions in Table 2, the estimates $\hat{n}$ seem to converge to the true root n as the number of Monte Carlo samples goes to infinity, provided that the root is unique. For cases of multiple roots, marked by triple asterisks in Table 2, the modified BIRA fails to find a root;

however, if the initial point is chosen so it is close to one of the roots, BIRA might converge to the nearest root. For example, when $\alpha = .99$, $\gamma = .001$, $k = .4125$ and the distribution shape is Johnson S_B with skewness 4 and kurtosis 30, Equation 3 has two roots 10 and 71 (see Figure 1(b)). If we choose the initial point N_0 as 50 and the initial number of Monte Carlo samples m_1 as 1000, then BIRA will return $\hat{n} = 71$ with standard error estimate 0.1. The computation time (evaluated by *nobs*) depends on, and usually increases with, the sample size n. The bigger the sample size, the greater the number of Monte Carlo observations generated from the distribution F_X, and therefore the longer the computation time.

4 ANALYSIS

This section is an extension of the sensitivity analysis for the tolerance factor k in Chen and Schmeiser (1995). The paper shows that k is an increasing function of α and of γ, but is not necessarily a monotonic function of n. Here we continue the analysis for (α, γ), (n, γ), and (n, α). We show that α is a decreasing function of γ, but α or γ is not necessarily monotonic with respect to n. Despite nonmonotonicity, when n goes to infinity, α converges to a constant $1 - F_X(\mu - k\sigma)$. Analogously, when n goes to infinity, γ converges to 1 if $k \geq k^\infty$ (recall $k^\infty = [\mu - F_X^{-1}(1-\alpha)]/\sigma$), and converges to 0, otherwise.

As in Chen and Schmeiser (1995), we use geometric graphs to illustrate the analysis. In the sample plane of $(S, \bar{X})$, define a straight line L as the set of sample points $(s, \bar{x})$ that satisfy $\bar{x} = ks + F_X^{-1}(1-\alpha)$. Then the geometric graph depends on the four tolerance parameters n, k, α, and γ, and the distribution shape as follows: (1) the spread of sample points $(s, \bar{x})$ depends on the sample size n and the distribution shape; (2) the slope of line L is k; (3) the $\bar{x}$-axis intercept $F_X^{-1}(1-\alpha)$ and s-axis intercept $-F_X^{-1}(1-\alpha)/k$ depend on α and the distribution shape; (4) the probability that a random point $(s, \bar{x})$ lies on or below line L is γ. We use these dependencies to analyze the (α, γ), (γ, n), and (α, n) interrelations in turn.

Figure 2 shows that the coverage α is a decreasing function of the confidence γ, given values of n, k, and the distribution shape. Fifty observations of $(S, \bar{X})$ from the Johnson S_B population with $\mu = 0$, $\sigma = 1$, skewness=4, kurtosis = 35, and sample size $n = 10$ are plotted. Two parallel lines, with the same slope $k = 1$, correspond to $\alpha = 0.7$ and 0.85. As α increases, the $\bar{x}$-axis intercept

Figure 2: The (α, γ) Relationship: Plot of Line L in the $(S, \bar{X})$ Sample Plane for Johnson S_B Distribution, $n = 10$, $k = 1$, and $\alpha = 0.7, 0.85$

decreases and the s-axis intercept increases, moving the line L parallel to the right. Therefore, γ, the probability of a point $(s, \bar{x})$ lying on or below L, decreases as α increases.

Figure 3: The (n, γ) Relationship: Plot of Line L in $(S, \bar{X})$ Sample Plane for Johnson S_B Distribution, $n = 10$, 300, $k = 0.5, 0.68, 1$, and $\alpha = 0.85$

Figure 3 shows that as the sample size n goes to infinity, the confidence γ nonmonotonically tends to 1 if $k \geq k^{\infty}$, and to 0, otherwise. The constant k^{∞} is the slope of the line joining the $\bar{x}$-axis intersect $(0, F_X^{-1}(1-\alpha))$ and the limiting point (σ, μ) of $(s, \bar{x})$. Fifty observations $(s, \bar{x})$, from the same population as in Figure 2, are plotted for $n = 10$ and $n = 300$. The three lines correspond to $\alpha = 0.85$ and $k = 0.5$, 0.68 $(= k^{\infty})$, and 1. When the sample size n goes to infinity, all sample points $(s, \bar{x})$ degenerate to the limiting point (σ, μ), which is $(1, 0)$ here. For line L with

$k = 1$ (greater than $k^{\infty} = 0.68$), the point (σ, μ) is below the line. Hence as n goes to infinity, the probability of lying on or below the line, i.e., γ, goes to 1. Similarly, for line L with $k = 0.5$ (less than k^{∞}), all points $(s, \bar{x})$ shrink to the point (σ, μ) above the line, as n goes to infinity, and hence γ goes to 0. The convergence of γ may not be monotonic however, even for the normal distribution. (See Figure 1).

Figure 4: The (n, α) Relationship: Plot of γ as a Function of n for $\alpha = 0.5, 0.55, 0.6$ in (a) and $\alpha = 0.8, 0.85, 0.9$ in (b), where $k = 0.5$, $\alpha^{\infty} = 0.66$, and the Distribution Shape is Johnson S_B

Finally we show that the coverage α converges to the constant $\alpha^{\infty} = 1 - F_X(\mu - k\sigma)$ as n goes to infinity, given values of k and γ, and the distribution shape. As discussed in Section 3.1, α is the $(1-\gamma)$th quantile of the random variable $C = \Pr_X\{X \geq \bar{X} - kS\}$. When n goes to infinity, the sample mean $\bar{X}$ and sample standard deviation S degenerate to μ and σ, respectively. Therefore the random variable C, and every quantile, converge to $\alpha^{\infty} = \Pr_X\{X \geq \mu - k\sigma\}$. Notice that α^{∞} depends on k and the distribution shape but not μ or σ. As for γ, coverage α may not converge monotonically unless k is a monotonic function of n.

For cases that the monotonicity holds, Figures 4(a) and 4(b), respectively, show that α increases with n for $\alpha \in (0, \alpha^{\infty}]$ and decreases with n, otherwise; in both situations, α converges to α^{∞}. In Figure 4(a), three curves illustrate that γ is an increasing function of n and converges to 1 for $\alpha = 0.5, 0.55, 0.6$, $k = 0.5$, and the Johnson S_B distribution with skewness 4 and kurtosis 35. The three α values are less than α^{∞} (0.66 here), therefore k must be greater than their associate k^{∞} values (recall that $\alpha^{\infty} = 1 - F_X(\mu - k\sigma)$ and $k^{\infty} = [\mu - F_X^{-1}(1-\alpha)]/\sigma)$, and hence the three curves increase monotonically to $\gamma = 1$. The two line segments E_1 and E_2 correspond to $n = 7$

and $\gamma = 0.75$, respectively. Since α is decreasing with γ, the intersections of the segment E_1 and the three curves, from top to bottom, correspond to the three increasing α values $0.5, 0.55, 0.6$. Furthermore, the intersections of the segment E_2 and the three curves, from left to right, illustrate that α increases as n increases. In the limit, α converges to α^{∞}. Similarly, Figure 4(b) shows that α decreases with n, converging to α^{∞} for $\alpha \in (\alpha^{\infty}, 1)$. The three curves illustrate that γ decreases to 0 as n goes to infinity for $\alpha = 0.8$, 0.85, 0.9 (larger than α^{∞}), where k and the distribution shape are as in Figure 4(a). The intersections of the line segment E_1 (corresponding to $n = 12$) and the three curves illustrate three increasing α values 0.8, 0.85, 0.9, from top to bottom. Therefore the intersections of the line segment E_2 (corresponding to $\gamma = 0.125$) and the three curves illustrate that α decreases as n increases; in the limit, α converges to α^{∞}.

5 AN EXAMPLE

A rocket-manufacturing engineer needs to build a reliability program so that with 0.99 confidence the engineer can state that the system reliability is at least 0.99. Suppose the testing plan is: Accept the rocket if $X \geq \bar{X} - kS$, based on a sample of size n. To operate the testing procedure, the sample size n and the tolerance factor k need to be chosen. The sample-size determination procedure is used to set the sample size. Also, from collected data on product characteristics, the characteristic is assumed to have a Johnson S_B distribution with skewness 4 and Kurtosis 30; the mean or variance is unknown. Given the lower specification limit $L = 0$ and a nominal value of $k = 1.938$, the sample-size procedure is as follows:

1. Compute the sample size n satisfying Equation 3, with $\alpha = .99$, $\gamma = .99$, $k = 1.938$, and the Johnson S_B distribution with $(\alpha_3, \alpha_4) = (4, 30)$. Table 2 gives $n = 10$.
2. Collect a sample $\{x_1, ..., x_n\}$ from the system. Compute $\bar{x} = 21$, $s = 10$, and then $k = (\bar{x} - L)/s = 2.1$.
3. Compute the coverage $\tilde{\alpha}$ satisfying Equation 2 with $n = 10$, $k = 2.1$, and $\gamma = .99$. The computed $\tilde{\alpha}$ is approximately 1.
4. Since $\tilde{\alpha} > 0.99$, stop and return $n = 10$.

With the sample size chosen as 10, $\alpha = 0.99$, $\gamma = 0.99$, and the distribution shape as Johnson S_B, the tolerance factor is then computed as $k = 1.938$ (Table 2). The values of n and k are then used in the testing plan.

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APPENDIX: JOHNSON DISTRIBUTION FAMILY

The Johnson family, proposed by Johnson (1949), includes three transformations of the standard normal distribution. Let X and Z denote the Johnson and standard normal random variables, respectively. The three transformations are:

$$
\begin{aligned}
S_L: &\quad Z = \eta + \delta \ln(\frac{X-\xi}{\lambda}), &\quad \lambda(X-\xi) \geq 0,\\
S_B: &\quad Z = \eta + \delta \ln(\frac{X-\xi}{\xi+\lambda-X}), &\quad 0 \leq X-\xi \leq \lambda,\\
S_U: &\quad Z = \eta + \delta \sinh^{-1}(\frac{X-\xi}{\lambda}), &\quad -\infty < X < \infty.
\end{aligned}
$$

The constants ξ and λ are location and scale parameters, respectively; η and δ are the shape parameters. The second transformation, S_B, provides a bounded random variable X; the third transformation, S_U, results in an unbounded X. For lognormal distributions, S_L, the range is bounded below if $\lambda > 0$ and bounded above if $\lambda < 0$. We use the numerical routines of Hill et al. (1976) to find the Johnson distribution having desired moments μ, σ, α_3, and α_4.

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Table 2: Sample-Size Estimation for the Johnson S_B Distribution with Skewness 4 and Kurtosis 30, and for the Johnson S_U Distribution with Skewness 2 and Kurtosis 30

Johnson S_B with $(\alpha_3, \alpha_4) = (4, 30)$							**Johnson S_U with $(\alpha_3, \alpha_4) = (2, 30)$**						
α	γ	k	n	$\hat{n}$	$\hat{se}(\hat{n})$	$nobs$ (1000's)	α	γ	k	n	$\hat{n}$	$\hat{se}(\hat{n})$	$nobs$ (1000's)
.001	.001	-19000	2	2	0	23	.001	.001	-7900	2	2	0	23
.001	.001	-74.9	10	9.1	.1	411	.001	.001	-32.1	10	9.2	.1	381
.001	.001	-32	30	27.1	.5	1589	.001	.001	-17.05	30	28.4	.2	1810
.001	.5	-27.4	2	2	0	12	.001	.5	-14.62	2	2	0	14
.001	.5	-12.6	10	10	.1	375	.001	.5	-8.94	10	10.1	.1	767
.001	.5	-10.3	30	29.4	.2	2090	.001	.5	-8.004	30	29.3	.2	3035
.001	.99	-1.5	2	2	0	165	.001	.99	-1.61	2	2	0	3066
.001	.99	-2.7	10	10.2	.1	2504	.001	.99	-2.56	10	9.5	.1	8291
.001	.99	-3.7	30	31.1	.1	4525	.001	.99	-3.31	30	30.1	.1	12332
.5	.001	-400	2	2	0	64	.5	.001	-186	2	2	0	7
.5	.001	-1.7	10	9.7	.1	380	.5	.001	-1.24	10	9.9	.1	539
.5	.001	-.24	30	29.8	.1	1272	.5	.001	-0.536	30	26.5	.4	2294
.5	.5	.3	2	2.2	.1	454	.5	.5	.072	2	2.3	.1	2645
.5	.5	.3563	10	***			.5	.5	.105	10	***		
.5	.5	.35	30	***			.5	.5	.1061	30	***		
.5	.99	15	2	2	0	14	.5	.99	17.7	2	2.1	.1	11
.5	.99	.89	10	10	0	234	.5	.99	.838	10	10	0	225
.5	.99	.62	30	30.5	.1	915	.5	.99	.49	30	30	0	1082
.99	.001	.43	2	2	0	13	.99	.001	-.18	2	2.1	.1	28
.99	.001	.4125	10	***			.99	.001	.56	10	***		
.99	.001	.38	30	***			.99	.001	.786	30	***		
.99	.5	1.35	2	2	0	20	.99	.5	4.34	2	2	0	22
.99	.5	.98	10	10	0	214	.99	.5	2.751	10	10.2	.1	574
.99	.5	.86	30	29.5	.2	696	.99	.5	2.49	30	32.6	.4	3021
.99	.99	67	2	2	0	23	.99	.99	242	2	2.2	.1	17
.99	.99	1.938	10	10.2	.1	216	.99	.99	6.71	10	10	0	188
.99	.99	1.34	30	30.3	.1	546	.99	.99	4.245	30	29.3	.1	942